Adventures in Supersingularland: An Exploration of Supersingular Elliptic Curve Isogeny Graphs

Sarah Arpin
University of Colorado Boulder
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Joint work with Catalina Camacho-Navarro, Kristin Lauter, Joelle Lim, Kristina Nelson, Travis Scholl, Jana Sotáková. [ACL^{+}19]

Overview

Motivation

Post-Quantum Cryptography

- NIST: 2015 call for proposals of post-quantum safe cryptography protocols
- Supersingular Isogeny Graph Cryptograhy: $\sim$ 15 years old: original hash function by Charles-Goren-Lauter [CGL06]; SIKE key exchange [Jao]

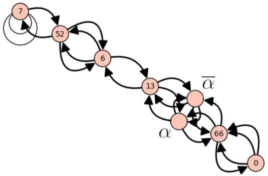
Hard Problems

- Path-finding in supersingular ℓ-isogeny graph
- Endomorphism ring computation [EHL$^+$18]

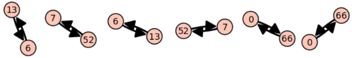
Three Graphs

- $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$:
  - Vertices: $\overline{\mathbb{F}}_p$-isomorphism classes of elliptic curves
  - Edges: ℓ-isogenies, up to equivalence
- $\mathcal{G}_\ell(\mathbb{F}_p)$:
  - Vertices: $\mathbb{F}_p$-isomorphism classes of elliptic curves
  - Edges: ℓ-isogenies, up to $\mathbb{F}_p$-equivalence
- Spine $\mathcal{S}$:
  - Subgraph of $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$
  - Vertices: $\overline{\mathbb{F}}_p$-isomorphism classes of curves with $j \in \mathbb{F}_p$
  - Edges: ℓ-isogenies up to $\overline{\mathbb{F}}_p$-equivalence

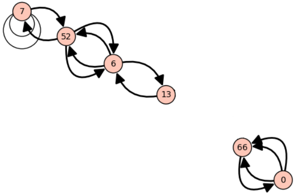
$\mathcal{G}_2(\overline{\mathbb{F}}_{89})$



$\mathcal{G}_2(\mathbb{F}_{89})$



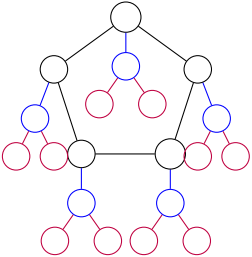
$\mathcal{S}$



Vertices labeled with j-invariants

$\mathcal{G}_\ell(\mathbb{F}_p)$: Volcanoes

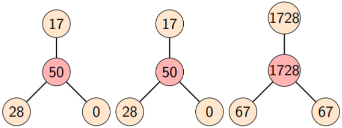
Ordinary ℓ-isogeny graphs



Kohel [Koh96];
Fouquet and Morain [FM02]

Supersingular ℓ-isogeny graphs $/\mathbb{F}_p$: p: a prime; E: supersingular elliptic curve over $\mathbb{F}_p$

$$\mathrm{End}_{\mathbb{F}_p}(E) \cong \begin{cases} \mathbb{Z}[\sqrt{-p}], & \text{floor} \\ \mathbb{Z}\left[\frac{1+\sqrt{-p}}{2}\right], & \text{surface} \end{cases}$$

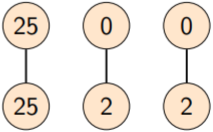


Structure of $\mathcal{G}_2(\mathbb{F}_p)$

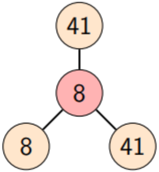
Delfs and Galbraith determined the structure of $\mathcal{G}_\ell(\mathbb{F}_p)$ [DG16].
For $\ell = 2$:

Theorem (Theorem 2.7 [DG16])

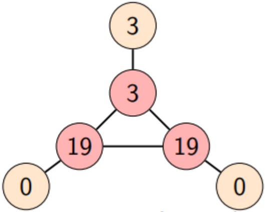
- $p \equiv 1 \pmod 4$*: Vertices paired together in isolated edges.*
- $p \equiv 3 \pmod 8$*: Vertices form volcanoes, each with four vertices: surface is one vertex connected to three vertices on the floor.*
- $p \equiv 7 \pmod 8$*: Vertices form a volcano; each surface vertex is connected 1:1 with the floor.*



$p = 29 \equiv 1 \pmod 4$



$p = 43 \equiv 3 \pmod 8$



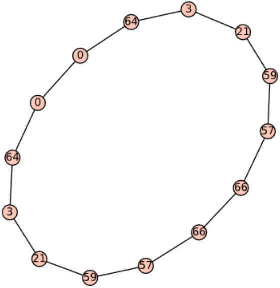
$p = 23 \equiv 7 \pmod 8$

Structure of $\mathcal{G}_\ell(\mathbb{F}_p)$

For $\ell > 2$:

Theorem (Theorem 2.7 [DG16])

- $\left(\frac{-p}{\ell}\right) = 1$: *two ℓ-isogenies*
- $\left(\frac{-p}{\ell}\right) = -1$: *no ℓ-isogenies*

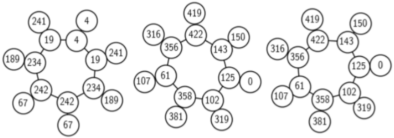


$p = 103$, $\ell = 3$

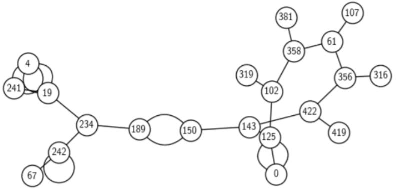
Possible changes, passing from $\mathcal{G}_\ell(\mathbb{F}_p)$ to $\overline{\mathbb{F}}_p$

Definition (3.13 ACL+19)

- If two distinct components of $\mathcal{G}_\ell(\mathbb{F}_p)$ have exactly the same set of vertices up to j-invariant, then they will **stack** over $\overline{\mathbb{F}}_p$.
- A component of $\mathcal{G}_\ell(\mathbb{F}_p)$ will **fold** if it contains both vertices corresponding to each j-invariant in its vertex set.
- Two distinct components of $\mathcal{G}_\ell(\mathbb{F}_p)$ will **attach with a new edge**.
- Two distinct components of $\mathcal{G}_\ell(\mathbb{F}_p)$ will **attach along a j-invariant** if one vertex of each share a j-invariant (only possible for $\ell > 2$).



(a) The $\mathcal{G}_2(\mathbb{F}_p)$ for $p = 431$



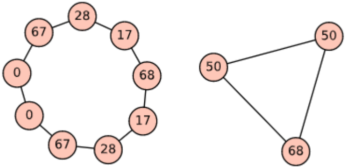
(b) The spine $\mathcal{S} \subset \mathcal{G}_2(\overline{\mathbb{F}}_p)$ for $p = 431$.

What actually happens for $\ell > 2$?

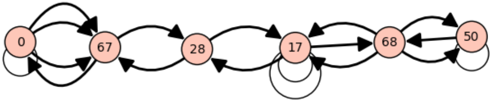
Theorem (Proposition 3.9 ACL+19)

Mapping $\mathcal{G}_\ell(\mathbb{F}_p)$ to $\mathcal{S}$, the only possible events are stacking, folding and n attachments by a new edge and m attachments along a j-invariant with $m + 2n \leq 2\ell(2\ell - 1)$.

$\mathcal{G}_3(\mathbb{F}_{83})$:



$\mathcal{S}$:

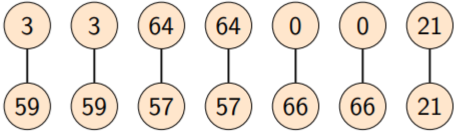


What actually happens for $\ell = 2$?

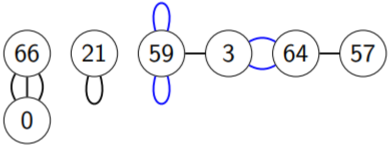
Theorem (Theorem 3.26 of ACL+19)

Mapping $\mathcal{G}_2(\mathbb{F}_p)$ to $\mathcal{S}$, only stacking, folding or at most one attachment by a new (double) edge are possible. No attachments by a j-invariant.

$\mathcal{G}_2(\mathbb{F}_{101})$:



$\mathcal{S}$:



Frobenius and Mirror Involution

$$E : y^2 = x^3 + ax + b \xrightarrow{Frob} E^{(p)} : y^2 = x^3 + a^p x + b^p$$
$$(x, y) \mapsto (x^p, y^p)$$
$$j(E) \mapsto j(E)^p$$

For $\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, let $\overline{\alpha}$ denote the Frobenius conjugate of α.
If α is supersingular, so is $\overline{\alpha}$.

Definition (Mirror Involution on $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$)

If $\exists$ ℓ-isogeny $\phi : E(\alpha_1) \to E(\alpha_2)$ then $\exists$ ℓ-isogeny $\phi' : E(\overline{\alpha}_1) \to E(\overline{\alpha}_2)$.

Given a path in $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$:

$$\begin{tikzcd} \alpha \arrow[r, dash] & \beta \arrow[r, dash] & \gamma \end{tikzcd}$$

Mirror Involution gives another path:

$$\begin{tikzcd} \overline{\alpha} \arrow[r, dash] & \overline{\beta} \arrow[r, dash] & \overline{\gamma} \end{tikzcd}$$

Mirror Paths

When can we connect a path with its mirror involution?

- α_i: j-invariants in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$
- a: j-invariant in $\mathbb{F}_p$

Option 1: Through an $\mathbb{F}_p$ vertex

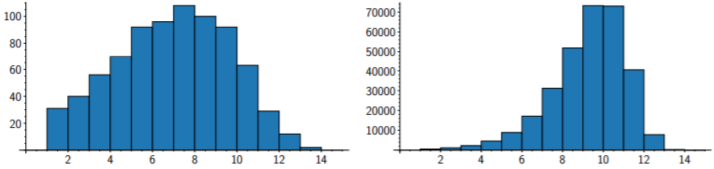
$$\alpha_0 \text{ --- } \alpha_1 \text{ --- } a \text{ --- } \overline{\alpha}_1 \text{ --- } \overline{\alpha}_0$$

Option 2: Through an ℓ-isogenous pair of conjugate vertices

$$\alpha_0 \text{ --- } \alpha_1 \text{ --- } \alpha_2 \text{ --- } \overline{\alpha}_2 \text{ --- } \overline{\alpha}_1 \text{ --- } \overline{\alpha}_0$$

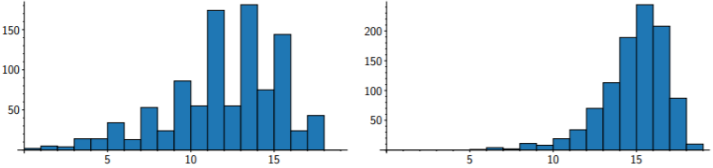
How often are paths of the first type? Second type?

How far are conjugate j-invariants in $\mathcal{G}_2(\overline{\mathbb{F}}_p)$?



(a) Distances between conjugate pairs. *(b) Distances between arbitrary pairs.*

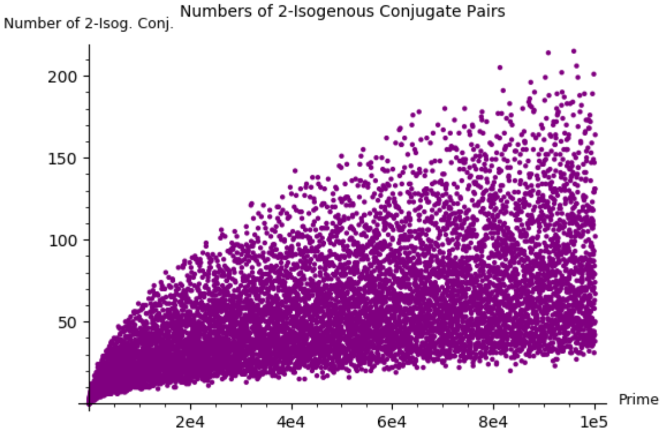
Figure 4.1: *Distances measured between conjugate pairs and arbitrary pairs of vertices not in $\mathbb{F}_p$ for the prime $p = 19489$.*



(a) Distances between conjugate pairs. *(b) Distances between arbitrary pairs.*

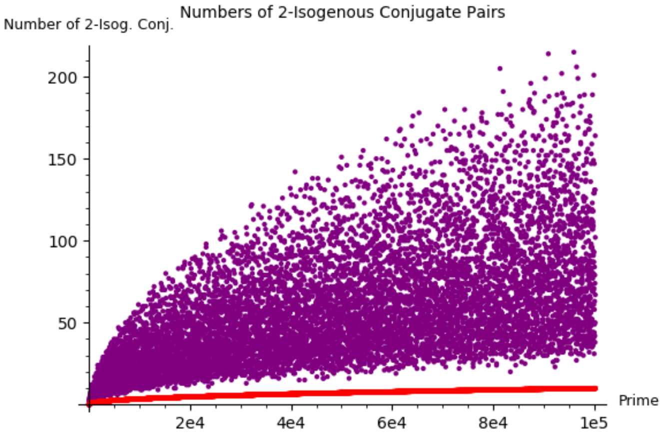
Figure 4.2: *Distances between 1000 randomly sampled pairs of arbitrary and conjugate vertices for the prime $p = 1000003$.*

How often are conjugate j-invariants 2-isogenous?



How often are conjugate j-invariants 2-isogenous?

[EHL+20]: Lower-bound on number of ℓ-isogenous conjugate j-invariants



Summary

- We understand completely how to map $\mathcal{G}_\ell(\mathbb{F}_p)$ into $\mathcal{G}_\ell(\overline{\mathbb{F}}_p)$.
- Mirror involution gives a new perspective on supersingular isogeny graph structure, further studied in [EHL$^+$20].
- Vertices which are conjugate appear to be closer than random vertices.
- Further heuristics on other interesting graph aspects can be found in our paper.

Thank you.

Sarah Arpin, Catalina Camacho-Navarro, Kristin Lauter, Joelle Lim, Kristina Nelson, Travis Scholl, and Jana Sotáková.
Adventures in Supersingularland.
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