Math 2300: Calculus Fall 2019

Lecture 37: Fall 2019

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One of each test! The instructions are the same for every problem: determine whether or not the series converges or diverges and prove your result.

37.1 Divergence Test

$$\sum_{n=1}^{\infty} \frac{3ne^n}{n^2+5}$$

$\lim_{n\to\infty} \frac{3ne^n}{n^2+5} = \frac{\infty}{\infty} \to$ l'Hopitals:

$= \lim_{n\to\infty} \frac{3e^n + 3ne^n}{2n} = \frac{\infty}{\infty} \to$ l'Hopital's: $= \lim_{n\to\infty} \frac{3e^n + 3e^n + 3ne^n}{2} = \infty \neq 0$

By the divergence test, the series diverges.

37.2 Geometric Series

*If this one converges, find the value of the sum.

$$\sum_{k=1}^{\infty} \frac{3^{2k-2}}{2^{3k+1}} = \sum_{k=1}^{\infty} \frac{(3^2)^k \cdot 3^{-2}}{(2^3)^k \cdot 2}$$

$$= \sum_{k=1}^{\infty} \frac{1}{18} \cdot \left(\frac{9}{8}\right)^k$$

diverges, b/c geometric series w/ $|r| = \frac{9}{8} \geq 1$.

37.3 Telescoping Series

*If this one converges, find the value of the sum.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{(n+1)(n+3)}}_{\text{partial fractions:}}$$

distribute

$$\frac{A}{n+1} + \frac{B}{n+3} = \frac{1}{(n+1)(n+3)} \quad \times (n+1)(n+3)$$

$$A(n+3) + B(n+1) = 1$$

$$n(A+B) + 3A + B = 1$$

$$\to A+B=0 \qquad 3A+B=1$$

$$A=-B \qquad -3B+B=1$$

$$-2B=1$$

$$B=\frac{-1}{2}$$

$$A=\frac{1}{2}$$

Telescoping Series:

$$\sum_{n=1}^{\infty}\left(\frac{1/2}{n+1} - \frac{1/2}{n+3}\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\left(\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots\right.$$

$$\left.\cdots + \left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N+1}}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+2}\right) + \left(\cancel{\frac{1}{N+1}} - \frac{1}{N+3}\right)\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{2} + \frac{1}{3} - \underbrace{\frac{1}{N+2}}_{\to 0} - \underbrace{\frac{1}{N+3}}_{\to 0}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{5}{6}\right) = \boxed{\frac{5}{12}}$$

Converges to ↑.

37.4 p-Series Test

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^4}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{4/3}}$$

Converges by p-series test w/ $p = 4/3$

37.5 Integral Test

$$\sum_{k=2}^{\infty} \frac{ln(k^4)}{k}$$ → consider:

$$f(x) = \frac{\ln(x^4)}{x} = 4\ln(x)x^{-1}$$

hypotheses:

① $f(x)$ is positive for $x > 1$ ✓

② $f(x)$ is continuous for $x > 0$ ✓

③ $f(x)$ is decreasing: show $f'(x) < 0$ for all x, eventually:

$$f'(x) = 4 \cdot \frac{1}{x} \cdot x^{-1} + 4\ln(x)(-1)x^{-2}$$

$$= \frac{4 - 4\ln(x)}{x^2} < 0 \text{ when}$$

$4 - 4\ln(x) < 0$, so $4(1 - \ln(x)) < 0$ when $x > e$ ✓

$$\int_2^{\infty} \frac{\ln(x^4)}{x}\,dx = \int_2^{\infty} \frac{4\ln(x)}{x}\,dx$$

$$= 4 \lim_{N\to\infty} \int_2^N \frac{\ln(x)}{x}\,dx \qquad u = \ln(x), \quad du = \frac{1}{x}dx$$

$$= 4 \lim_{N\to\infty} \int_{x=2}^{x=N} u\,du$$

$$= 4 \lim_{N\to\infty} \left(\frac{u^2}{2} \Big|_{x=2}^{x=N} \right)$$

$$= 4 \lim_{N\to\infty} \left(\frac{(\ln(N))^2}{2} - \frac{(\ln(2))^2}{2} \right) = \infty$$

By the integral test, the series diverges.

37.6 Comparison Test

num is pos. for $n \geq 1$, denom is pos for $n \geq 1$ ⇒ positive ✓

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$ compare to $\frac{1}{n}$ (positive for $n \geq 1$)

$$\frac{n}{n^2 - \cos^2(n)} \geq \frac{n}{n^2} = \frac{1}{n}, \text{ since } \cos^2(n) \in [0,1]$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, since it's the harmonic series

→ $\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$ diverges by the comparison test: its terms are $\geq$ those of a divergent series

37.7 Limit Comparison Test

$$\sum_{n=2}^{\infty} \frac{4n^2+n}{\underbrace{\sqrt[3]{n^7+n^3}}_{\text{positive, since } n \geq 2}} \quad \text{compare to } \frac{n^2}{n^{7/3}} = \frac{1}{n^{1/3}}$$

positive terms, and $\sum_{n=2}^{\infty} \frac{1}{n^{1/3}}$ diverges by p-series test w/ $p = 1/3 < 1$.

$$\lim_{n\to\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}} \cdot \frac{n^{1/3}}{1} = \lim_{n\to\infty} \frac{4n^{7/3}+n^{4/3}}{\sqrt[3]{n^7+n^3}} = 4 \leftarrow \text{positive, finite constant}$$

By the limit comparison test, $\sum_{n=2}^{\infty} \frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$ diverges.

37.8 Alternating Series Test

$$\sum_{n=0}^{\infty} \frac{\cos(\pi n)n}{2^n+3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n+3^n}$$

$|a_n| = \frac{n}{2^n+3^n}$ ← to see decreasing, look at $f(x) = \frac{x}{2^x+3^x}$:

$$f'(x) = \frac{(2^x+3^x) - x(2^x\ln(2)+3^x\ln(3))}{(2^x+3^x)^2} \leftarrow \text{always positive.}$$

numerator is negative for $x \geq 2 \rightarrow f'(x) < 0$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{2^n+3^n} = \frac{\infty}{\infty} \rightarrow \text{l'Hopital:}$$

$$= \lim_{n\to\infty} \frac{1}{2^n\ln(2)+3^n\ln(3)} = 0$$

$\therefore$ By the alternating series test, the series converges.

37.9 Ratio Test

$$\sum_{n=4}^{\infty} \frac{e^{4n}}{(n-2)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{e^{4(n+1)}}{(n+1-2)!}\cdot\frac{(n-2)!}{e^{4n}}$$

$$= \lim_{n\to\infty} \frac{e^{4n}\cdot e^4}{(n-1)!}\cdot\frac{(n-2)!}{e^{4n}}$$

$$= \lim_{n\to\infty} \frac{e^4}{(n-1)} = 0 < 1$$

$\Rightarrow$ By the ratio test, the series is absolutely convergent.

37.10 Absolute/Conditional Convergence

Determine if the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{j=0}^{\infty} \frac{(-3)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j (3)^j}{j!}$$

Use ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

By the ratio test, the series is absolutely convergent.