Math 2300: Quiz 7, 10/11/2019

Name: ______________________________ Score: ________

Please show your work on all questions.

1. (5 points) Does the following series converge or diverge? Prove your conclusion. If it converges, to what value? Be sure to carefully state the test you use to prove your conclusion.

$$\sum_{n=1}^{\infty} 9^{-n+2} \cdot 4^{n+1}$$

Solution:

$$\begin{aligned} \sum_{n=1}^{\infty} 9^{-n+2} \cdot 4^{n+1} &= \sum_{n=1}^{\infty} 9^{-n} 9^2 \cdot 4^n 4^1 \\ &= \sum_{n=1}^{\infty} 9^2 \cdot 4 \cdot \left(\frac{4}{9}\right)^n \\ &= \sum_{n=1}^{\infty} 9^2 \cdot 4 \cdot \left(\frac{4}{9}\right) \cdot \left(\frac{4}{9}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} 9 \cdot 4^2 \cdot \left(\frac{4}{9}\right)^{n-1} \end{aligned}$$

This is a convergent geometric series, since $|r| = \frac{4}{9} < 1$. It converges to:

$$\frac{a}{1-r} = \frac{9 \cdot 4^2}{1 - \frac{4}{9}} = \frac{9 \cdot 4^2}{\frac{5}{9}} = \frac{9 \cdot 4^2}{1} \cdot \frac{9}{5} = \frac{9^2 \cdot 4^2}{5}.$$

2. (5 points) Use the integral test to determine if the following series converges or diverges. *Hint: Be sure to show the hypotheses are satisfied!*

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

Solution:
To use the integral test, we need to show that $f(x) = \frac{\ln(x)}{x}$ is positive, continuous, and decreasing on an appropriate interval.

- $f(x) = \frac{\ln(x)}{x}$ is continuous for $x > 0$, so it is certainly continuous for $x \geq 1$.
- $f(x) = \frac{\ln(x)}{x} > 0$ for $x \geq 1$, since $\ln(x) \geq 0$ for $x \geq 1$ and the denominator is also positive on this interval.
- For decreasing, look at $f'(x) = \frac{x\frac{1}{x} - \ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$. This is negative when $1 - \ln(x)$ is negative, so for $x > e$. This means we are decreasing for $x \geq 3$. (Note: "Eventually decreasing" is good enough to use the integral test.)

Now, we compute the appropriate integral:

$$\int_1^\infty \frac{\ln(x)}{x} dx = \lim_{N\to\infty} \int_1^N \frac{\ln(x)}{x} dx$$

$$\text{u-sub: } u = \ln(x); du = \frac{1}{x} dx$$

$$= \lim_{N\to\infty} \int_{x=}^{x=N} u du$$

$$= \lim_{N\to\infty} \frac{u^2}{2} |_{x=}^{x=N}$$

$$= \lim_{N\to\infty} \frac{\ln(N)^2}{2} - \frac{\ln(1)^2}{2}$$

$$= \infty$$

By the integral comparison test, the series diverges.