Math 2300: Quiz 5, 9/27/2019

Name: _____ Score: _____

Please show your work on all questions. Note that there is a question on the back!
Due at the beginning of class Monday 9/30.

1. (4 points) A force of $F(x) = x^2 - e^{3x}$, with x in meters, acts on an object. What is the work required to move the object from $x = 3$ to $x = 7$?
   Solution:

$$\begin{aligned} W &= \int_3^7 F(x)dx \\ &= \int_3^7 (x^2 - e^{3x})dx \\ &= (\frac{x^3}{3} - \frac{e^{3x}}{3})_3^7 \\ &= (\frac{7^3}{3} - \frac{e^{3\cdot 7}}{3}) - (\frac{3^3}{3} - \frac{e^{3\cdot 3}}{3}) \\ &= \frac{343 - 27 - e^{21} + e^9}{3} \\ &= \frac{316 - e^{21} + e^9}{3} \end{aligned}$$

   No units on this one, because I forgot to give you units for force (F(x)).

2. (2 points) Find the average value of the function $f(x) = x^4\sqrt{1-x^2}$ over the interval from $x = 0$ to $x = 1$.

Solution:

#2

$$\int_{x=0}^{x=1} x^4\sqrt{1-x^2}\,dx$$

$\sin\theta = \sqrt{1-x^2}$ $\quad \cos\theta = x$

$\cos^4\theta = x^4$

$dx = -\sin\theta\,d\theta$

$$\int_{x=0}^{x=1} \cos^4(\theta)\sin\theta(-\sin\theta\,d\theta)$$

$$= -\int_{x=0}^{x=1} \cos^4(\theta)\sin^2\theta\,d\theta$$

power reduce $\quad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$ $\quad \sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$$= -\int_{x=0}^{x=1} \left[\frac{1}{2}(1+\cos(2\theta))\right]^2 \cdot \frac{1}{2}[1-\cos(2\theta)]\,d\theta$$

$$= -\frac{1}{8}\int_{x=0}^{x=1} (1+2\cos(2\theta)+\cos^2(2\theta))(1-\cos(2\theta))\,d\theta$$

$$= -\frac{1}{8}\int_{x=0}^{x=1} (1+\cos(2\theta)-\cos^2(2\theta)-\cos^3(2\theta))\,d\theta$$

$$= -\frac{1}{8}\int_{x=0}^{x=1} \left[1+\cos(2\theta)-\frac{1}{2}(1+\cos(4\theta))-(1-\sin^2(2\theta))\cos(2\theta)\right]d\theta$$

$$= -\frac{1}{8}\int_{x=0}^{x=1} \left[\frac{1}{2}+\cos(2\theta)-\frac{1}{2}\cos(4\theta)-\cos(2\theta)+\sin^2(2\theta)\cos(2\theta)\right]d\theta$$

$$= -\frac{1}{8}\left[\frac{1}{2}\theta-\frac{1}{8}\sin(4\theta)+\frac{1}{3}\sin^3(2\theta)\right]\Big|_{x=0}^{x=1}$$

$$= \left[-\frac{1}{16}\theta+\frac{1}{64}(2\sin(2\theta)\cos(2\theta))-\frac{1}{24}(2\sin(\theta)\cos(\theta))^3\right]_{x=0}^{x=1}$$

$$= \left[-\frac{1}{16}\cos^{-1}(x)+\frac{1}{64}(2(2\sin\theta\cos\theta)(\cos^2\theta-\sin^2\theta))-\frac{1}{3}(\sin^3(\theta)\cos^3(\theta))\right]\Big|_{x=0}^{x=1}$$

$$= \left[-\frac{1}{16}\cos^{-1}(x)+\frac{1}{16}(\sqrt{1-x^2})(x)(x^2-(1-x^2))-\frac{1}{3}((\sqrt{1-x^2})^3x^3)\right]\Big|_{x=0}^{x=1}$$

$$= \left[\frac{-1}{16}(0)+\frac{1}{16}(0)-\frac{1}{3}(0)\right]-\left[\frac{-1}{16}\left(\frac{\pi}{2}\right)+\frac{1}{16}(0)-\frac{1}{3}(0)\right]$$

$$= \boxed{\frac{\pi}{32}}$$

3. (4 points) Find the work done pumping molasses out of a 10 foot tall cylindrical tank with a radius of 2 feet. The density of molasses is $100lbs/(ft)^3$ (note that this includes gravity!), and the tank is completely full to begin with. Draw at least one picture to accompany your work.

Solution:

Note that the shape is a cylinder, so the cross sections are all circles of radius $r = 2$. Look at one slice first:

$$\begin{aligned} W_{\text{slice}} &= F * d \\ &= (\rho * vol) * d \\ &= (100 * \pi r^2 * dy) * (10 - y) \\ &= 400\pi(10 - y)dy \end{aligned}$$

Now integrate it:

$$\begin{aligned} W &= \int_0^{10} 400\pi(10 - y)dy \\ &= 400\pi(10y - \frac{y^2}{2})|_0^{10} \\ &= 400\pi(100 - 50) \\ &= 20000\pi \end{aligned}$$

So the solution is $20000\pi ft \cdot lbs$.