Math 2300: Calculus | Fall 2019

Lecture 21: Tuesday September 24 - Friday September 27

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21.1 6.6: Avg Value, Work, Hydrostatic Pressure, Center of Mass

21.1.1 AVG VALUE

1. Find the average value of the function $f(x) = t^2 - 5t + 6\cos(\pi t)$ over $[-1, \frac{5}{2}]$.

$f_{av} = \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$

$f_{av} = \frac{1}{\frac{5}{2}-1}\int_{-1}^{5/2}(t^2-5t+6\cos(\pi t))\,dt$

$= \frac{1}{7/2}\left[\frac{t^3}{3} - \frac{5t^2}{2} + \frac{6}{\pi}\sin(\pi t)\Big|_{-1}^{5/2}\right]$

$= \boxed{\frac{2}{7}\left[\frac{(5/2)^3}{3} - \frac{5(5/2)^2}{2} + \frac{6}{\pi}\sin\left(\frac{5\pi}{2}\right)\right] - \frac{2}{7}\left[\frac{-1}{3} - \frac{5}{2} + 0\right]}$

21.1.2 WORK

Work = F·d

$W = \int(\text{Force})\,dx$

$W = \int(\text{Work Slice})$

$W = \int(\text{Force slice} \times \text{dist.})$

1. A box is slid 3 meters across a carpet against a force of kinetic friction of 45N. How much work is done?

$W = F\cdot d = 3\cdot 45 = \boxed{135\ N\cdot m}$

2. I am pushing my sister across a 10 foot room. She pushes back with increasing ferocity, with a force of $20 + \frac{x^2}{2}$ pounds, where x is how far I have pushed her. How much work do I do?

$W = \int_0^{10}\left(20+\frac{x^2}{2}\right)dx = 20x + \frac{x^3}{6}\Big|_0^{10} = 20(10) + \frac{10^3}{6}$

$= \boxed{366.\overline{6}\ \text{ft}\cdot\text{lbs}}$

3. A 30 cm long spring with a spring constant of $k = 120$ N/m is compressed to 20 cm. Calculate the work done.

$F = kx = 120x$ compression from 0 to 10cm

$\rightarrow$ 10cm = 0.1m

$\int_0^{.1} 120x\,dx = 60x^2\Big|_0^{0.1} = \boxed{0.6\ N}$

4. A force of 10 lbs is required to hold a spring stretched to 6 inches past its natural length. Calculate the work required to stretch it 8 inches past its natural length.

.5 ft.

2/3 ft.

$F = kx$ first find k:

$10 = k(.5)$

$20 = k$

$\Rightarrow F = 20x$

$\rightarrow W = \int_0^{2/3} (20x)\,dx = 10x^2 \Big|_0^{2/3} = \boxed{\frac{40}{9} \text{ ft}\cdot\text{lbs}}$

5. How much energy is required to hoist a 3-kilogram pumpkin 15 meters to the roof of the math building?

$F = m \cdot g$

$F = 3(9.8)$

★ Force is constant whole way $\rightarrow$ $W = Fd = 3(9.8)(15)$

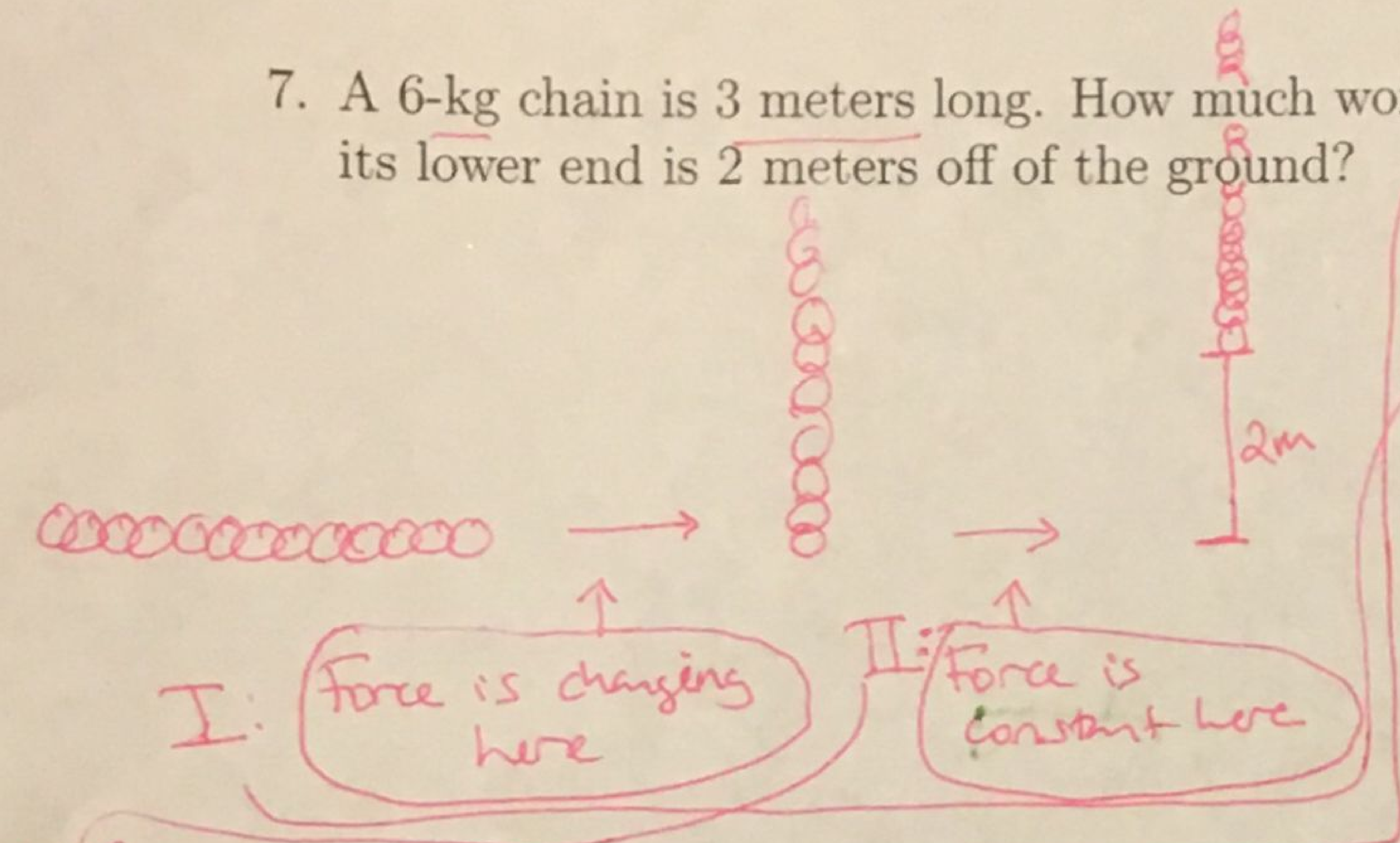
$= \boxed{441 \text{ N}\cdot\text{m}}$

6. How much energy is required to carry a 44-lb stack of books up to the third floor of the math building? (30 ft.)

$F = 44$ ★ Force is constant whole way

$\rightarrow W = Fd = 44 \cdot 30 = \boxed{1320 \text{ ft}\cdot\text{lbs}}$

7. A 6-kg chain is 3 meters long. How much work is done lifting it from the ground until its lower end is 2 meters off of the ground?



I: $F = m \cdot g$

mass depends on how far it has been lifted so far. If it has been lifted x meters, the mass is $6 \cdot \frac{x}{3}$, since we have the fraction $\frac{x}{3}$ of the weight in our hands.

$\rightarrow F = (2x)\cdot(9.8)$

$$W = \int_0^3 (2x)(9.8)\,dx = 19.6\int_0^3 x\,dx = 9.8x^2\Big|_0^3 = 88.2\text{ J}$$

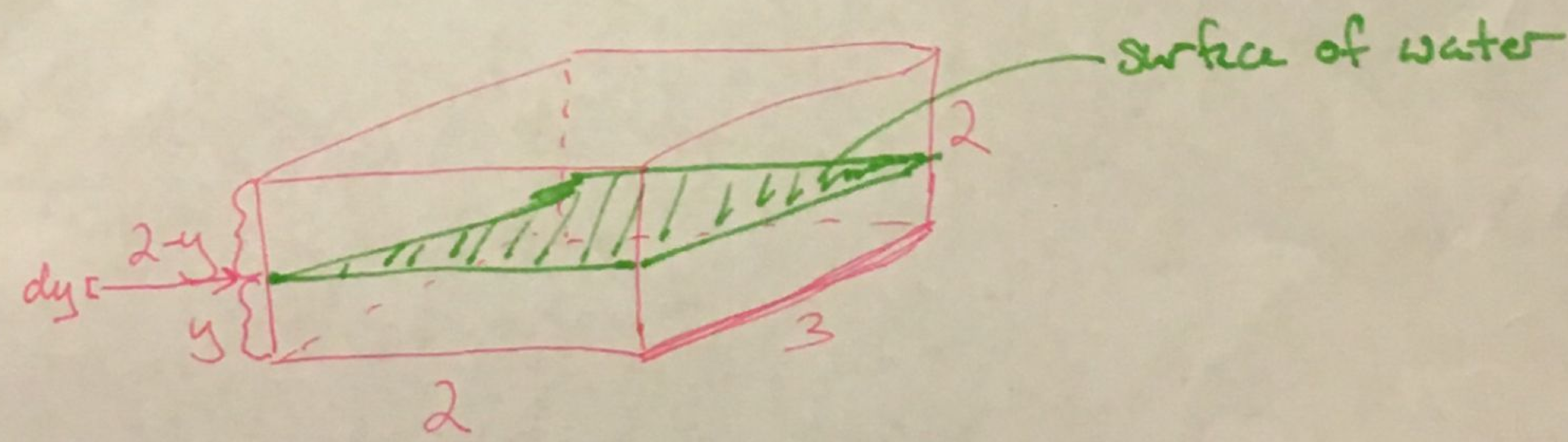
II: $F = mg = 6 \cdot 9.8 = 58.8$

$\rightarrow W = Fd = 58.8 \cdot 2 = 117.6\text{ J}$

I + II $= 88.2 + 117.6$ J

$= \boxed{205.8\text{ J}}$

8. How much work is done emptying a $2 \times 2 \times 3$-ft rectangular tank? The water must be pumped to a point in the upper corner of the tank.



Force = mass $\cdot$ d

★ mass in this case depends on the density of the water

mass = volume $\cdot \rho$, w/ ρ = density of water $= 62.5 \frac{\text{lbs}}{\text{ft}^3}$

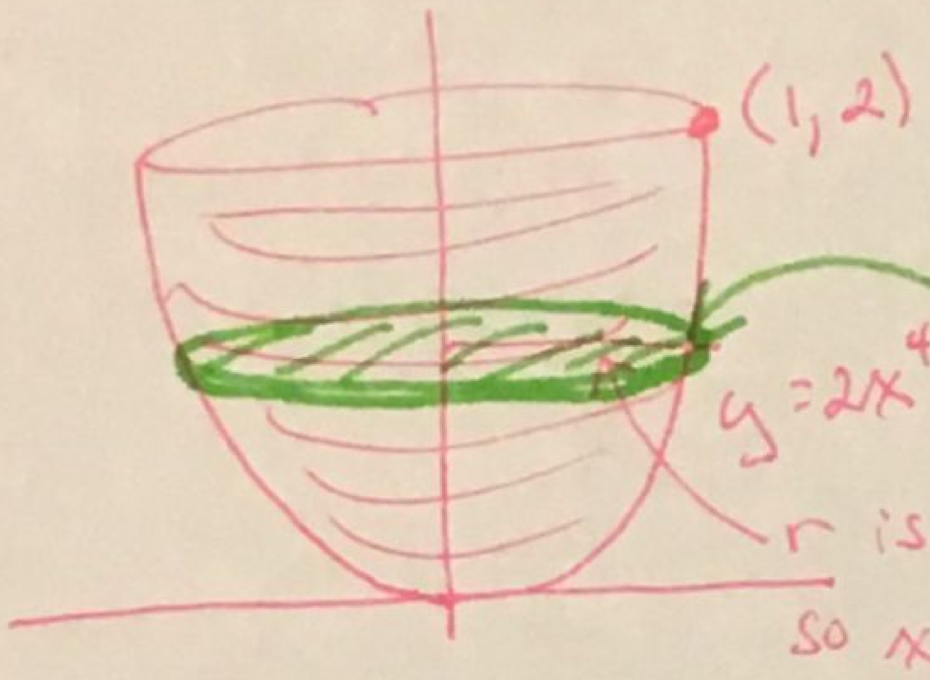
volume of a "slice" is Area $\times dy = 2 \times 3 \times dy = 6dy$

$\rightarrow$ $\underset{\text{slice}}{\text{Force}} = (6dy) \times \rho = 375\,dy$

distance to travel when water is at height $y = (2-y)$

$$\Rightarrow W = \int_0^2 \underbrace{375(2-y)\,dy}_{\text{Work slice}} = 375\left(2y - \frac{y^2}{2}\right)\Big|_0^2 = 375(2) = \boxed{750\text{ ft}\cdot\text{lbs}}$$

9. A tub has the shape of the solid of revolution formed by rotating around the y-axis the portion of the curve $y = 2x^4$ that lies between $x = 0$ and $x = 1$. (Draw a picture.) How much work is done to empty the tank? All of the water must be pumped out of the top of the tank.



$W_{slice} = F_g \cdot d$

$= m \cdot g \cdot d$

$= V_{slice} \cdot \rho \cdot g\,(2-y)$ ★No units given so you choose.

$= \pi r^2 dy\, \rho \cdot (9.8)(2-y)$

$= \pi \sqrt{\frac{y}{2}}\, dy\, \rho \cdot (9.8)(2-y)$

$= \pi \sqrt{\frac{y}{2}} \cdot (98)(2-y)\,dy$

ρ = mass density of water

$= 10^3 \frac{kg}{m^3}$ and $g = 9.8 m/s^2$

or $= \underbrace{62.5 \frac{lbs}{ft^3}}_{\text{(includes gravity)}}$

so $= \rho \cdot g$

$\rightarrow W = \int_0^2 \frac{\pi \cdot 98}{\sqrt{2}} \cdot \sqrt{y}\,(2-y)\,dy$

$= \frac{98\pi}{\sqrt{2}} \int_0^2 \left(2y^{1/2} - y^{3/2}\right) dy$

$= \frac{98\pi}{\sqrt{2}} \left(\frac{4x^{3/2}}{3} - \frac{2y^{5/2}}{5}\right)\Big|_0^2$

$= \frac{98\pi}{\sqrt{2}} \left(\frac{4\sqrt{8}}{3} - \frac{2\sqrt{32}}{5}\right)$

$= 98\pi \left(\frac{8}{3} - \frac{8}{5}\right)$

$= \boxed{\frac{98\pi \cdot 16\,J}{15}}$

21.1.3 HYDROSTATIC PRESSURE + Force

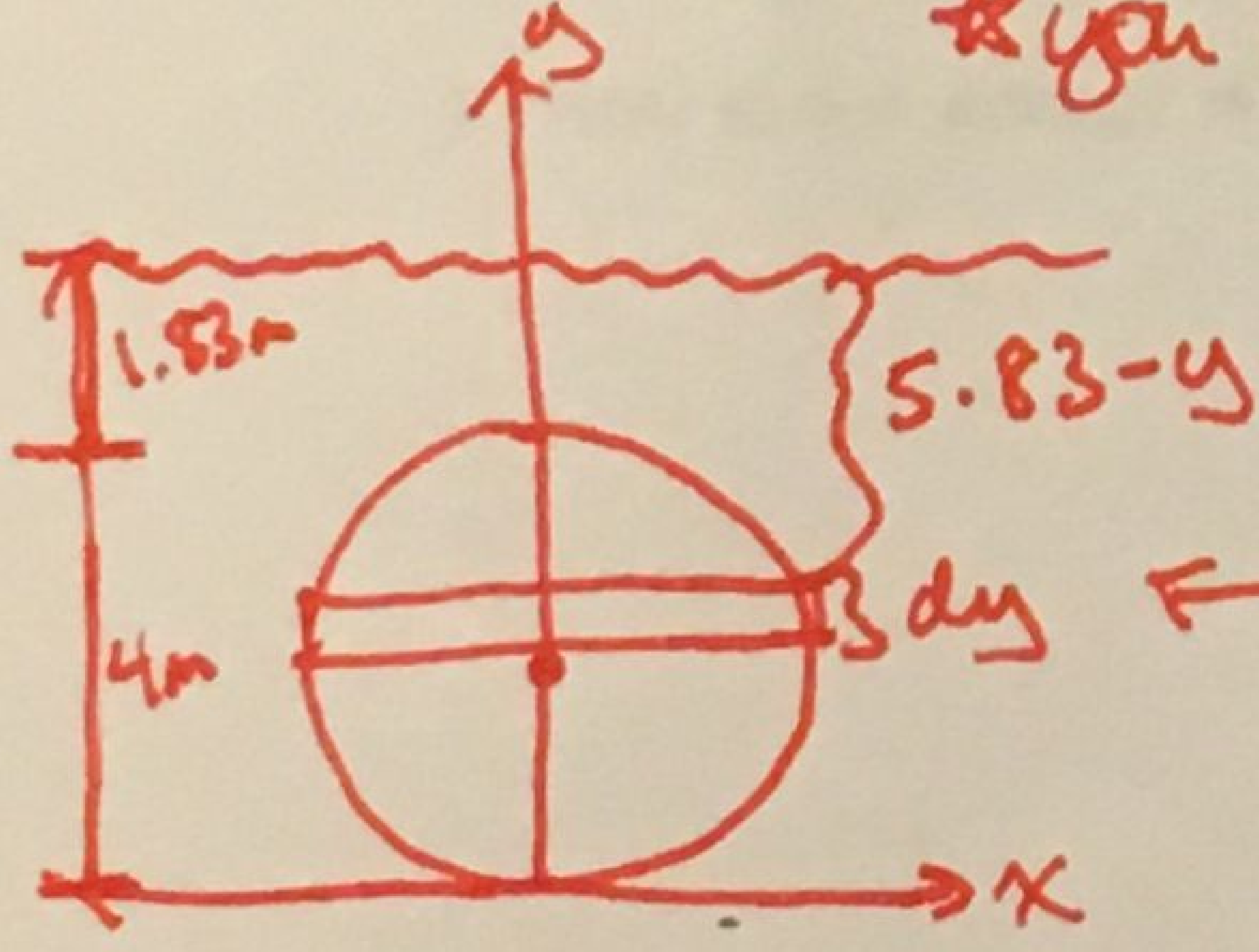
1. A circular plate of radius 2m is submerged 6 feet deep (measured from the top of the plate). Find the hydrostatic force on the plate.

Hydrostatic Force is pressure acting on area : $F = P \cdot A$

And $P = \rho \cdot g \cdot d$, where d=depth.

6ft = 1.83 meters

* you choose how to setup your coordinates! Draw a picture.



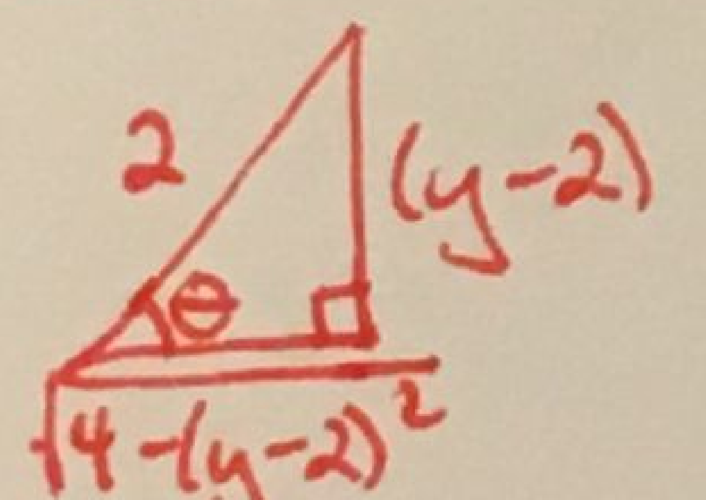
$\leftarrow$ width $= 2\sqrt{4-(y-2)^2}$

Eq. of circle $= (y-2)^2 + x^2 = 4$

$$H.F. = \int_0^4 \underbrace{\rho \cdot g \cdot (5.83-y)}_{\text{pressure}} \cdot \underbrace{2\sqrt{4-(y-2)^2} \cdot dy}_{\text{Area}}$$

$$= 2\rho g \int_0^4 (5.83-y)\sqrt{4-(y-2)^2}\, dy$$

Trig sub



$2\sin\theta = y-2$

$2\sin\theta + 2 = y$

$2\cos\theta\, d\theta = dy$

$2\cos\theta = \sqrt{4-(y-2)^2}$

$$= 2\rho g \int_{y=0}^{y=4} (5.83 - (2\sin\theta + 2)) \cdot 2\cos\theta \cdot 2\cos\theta\, d\theta$$

$$= 2\rho g(5.83)(4) \int_{y=0}^{y=4} \cos^2\theta\, d\theta - 2\rho g \cdot 8 \int_{y=0}^{y=4} \sin\theta \cos^2\theta\, d\theta$$

$$= 2\rho g(15.32) \int_{y=0}^{y=4} \tfrac{1}{2}(\cos(2\theta)+1)\, d\theta - 2\rho g \cdot 8 \left(-\tfrac{1}{3}\cos^3\theta \Big|_{y=0}^{y=4}\right)$$

$$= 15.32\rho g\left(\tfrac{1}{2}\sin(2\theta) + \theta \Big|_{y=0}^{y=4}\right) + \frac{16\rho g}{3}\left[\left(\frac{\sqrt{4-(y-2)^2}}{2}\right)^3 \Bigg|_{y=0}^{y=4}\right]$$

$$= 15.32\rho g\left(\cos\theta \sin\theta + \sin^{-1}\left(\tfrac{y-2}{2}\right) \Big|_{y=0}^{y=4}\right) + \frac{16\rho g}{3}[0-0]$$

$$= 15.32\rho g\left(\frac{\sqrt{4-(y-2)^2}}{2} \cdot \frac{y-2}{2} + \sin^{-1}\left(\frac{y-2}{2}\right)\right)\Bigg|_{y=0}^{y=4} = 15.32\rho g\pi \approx \boxed{472147.45\ J}$$

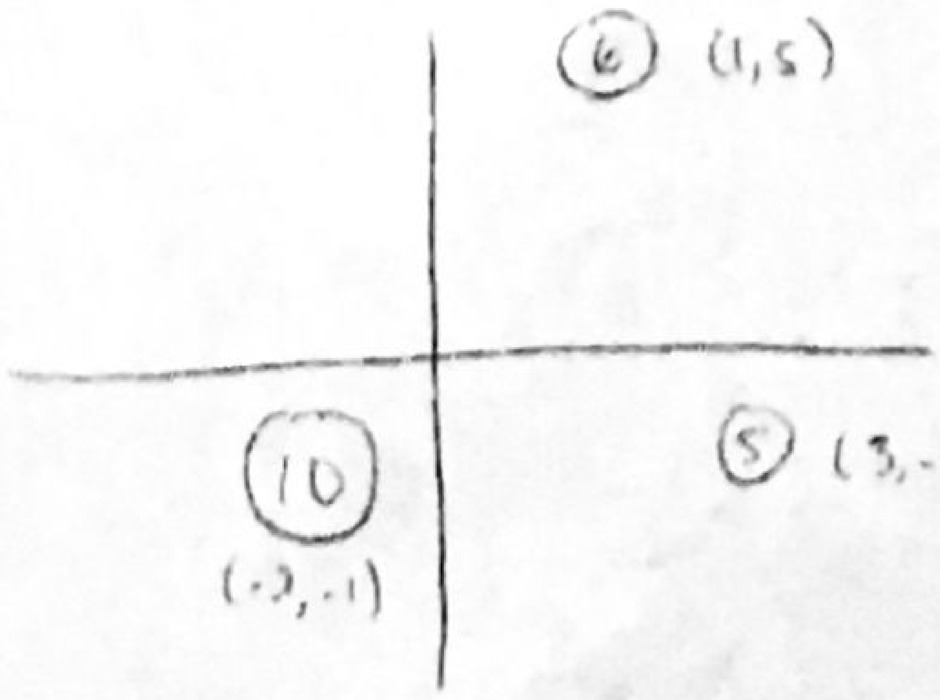
Examples:

1. Find the moments M_x and M_y and the center of mass of the system of the following point masses.

   - A mass of 6 at the point (1,5)
   - A mass of 5 at the point (3,-2)
   - A mass of 10 at the point (-2,-1)

$$M_x = \overbrace{6\cdot 5 + 5(-2) + 10(-1)}^{y\text{-coords}} = 10$$

$$M_y = \underbrace{6\cdot 1 + 5\cdot 3 + 10(-2)}_{x\text{-coords}} = 1$$

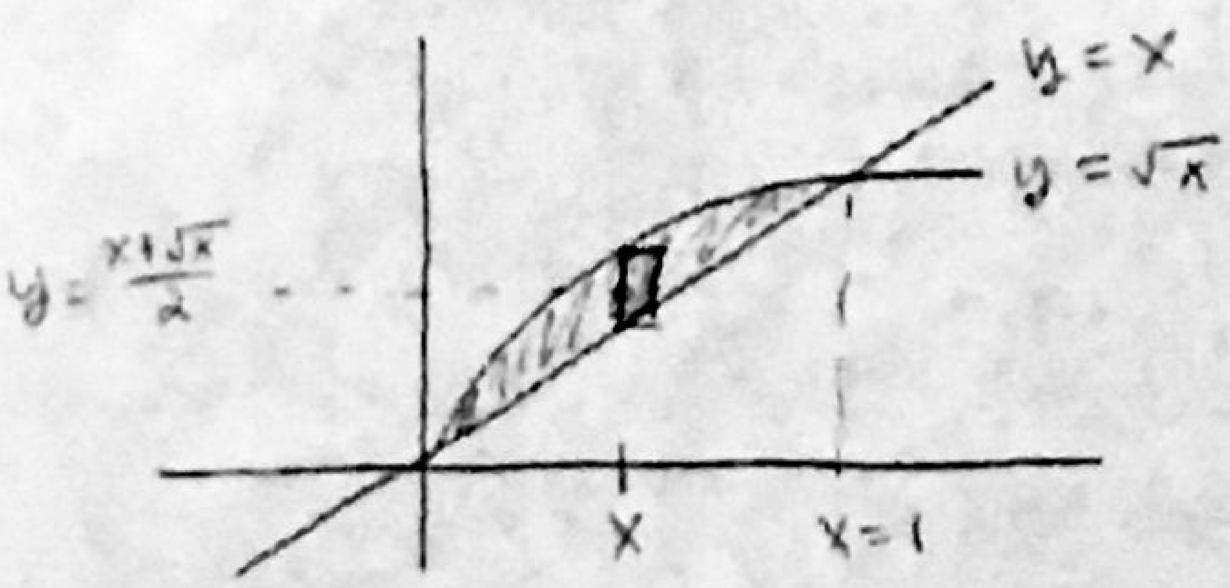


$$\bar{X} = \frac{M_y}{\text{total mass}} = \frac{1}{6+5+10} = \frac{1}{21}$$

$$\bar{Y} = \frac{M_x}{\text{total mass}} = \frac{10}{6+5+10} = \frac{10}{21}$$

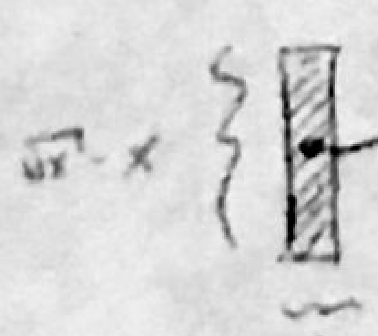
$$\boxed{\begin{array}{l} \text{COM}: \left(\frac{1}{21}, \frac{10}{21}\right) \\ M_x: 10 \\ M_y: 1 \end{array}}$$

2. Find the centroid of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.



$$\bar{X} = \frac{\int_0^1 \tilde{x}\cdot \text{mass}}{\int_0^1 \text{mass}} = \frac{\int_0^1 x(\sqrt{x}-x)\,dx}{\int_0^1 \sqrt{x}-x\,dx} = \cdots = \frac{1/15}{1/6}$$

$$= \frac{6}{15} = \frac{2}{5}$$



center $(\tilde{x}, \tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{x+\sqrt{x}}{2}$

mass $= \rho \cdot$ Area

$= 1\cdot(\sqrt{x}-x)\,dx$

(Assume $\rho = 1$. see #3 for explanation)

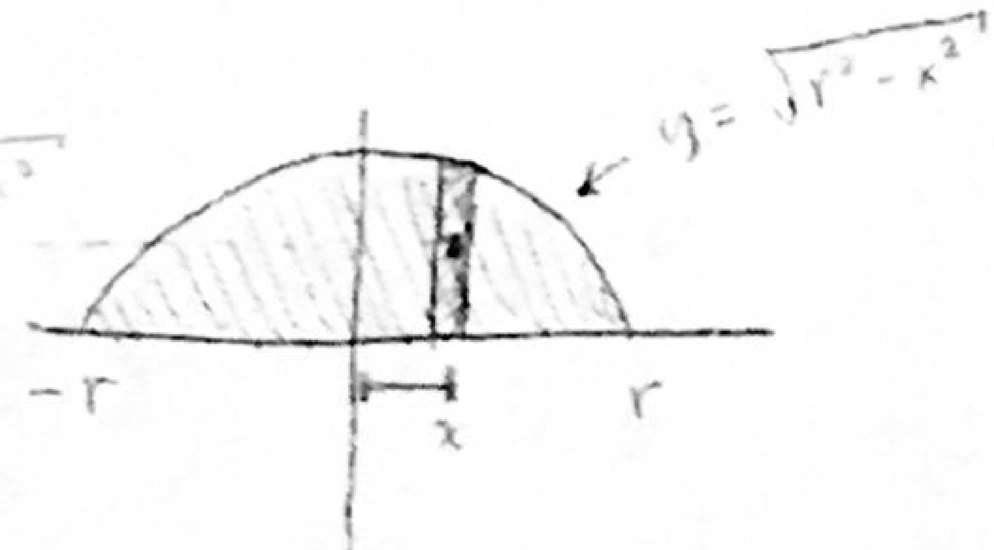
$$\bar{Y} = \frac{\int_0^1 \tilde{y}\cdot\text{mass}}{\int_0^1 \text{mass}} = \frac{\int_0^1 \frac{x+\sqrt{x}}{2}\cdot(\sqrt{x}-x)\,dx}{\int_0^1 \sqrt{x}-x\,dx} = \cdots = \frac{1/12}{1/6}$$

$$= \frac{6}{12} = \frac{1}{2}$$

$$\boxed{\text{COM}: \left(2/5, \tfrac{1}{2}\right)}$$

3. Find the center of mass of the semicircular plate of radius r.

Assume $\rho = 1$ because ρ "cancels" out in COM calculation. Note: need ρ for M_x or M_y, but we're not looking for that here



$$\bar{X} = \frac{\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx}{\int_{-r}^{r}\sqrt{r^2-x^2}\,dx} \leftarrow \frac{\pi r^2}{2} \text{ Area of semi-circle}$$

$$= \frac{2}{\pi r^2}\cdot\int_{-r}^{r} x\sqrt{r^2-x^2}\,dx \qquad \left\{\begin{matrix} u = r^2-x^2 \\ du = -2x\,dx \end{matrix}\right\}$$

$$= \frac{2}{\pi r^2}\cdot\frac{-1}{2}\int_{u=0}^{u=0}\sqrt{u}\,du$$

$$= \frac{-1}{\pi r^2}\cdot 0 = 0$$

$\sqrt{r^2-x^2}$

center $(\tilde{x},\tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{1}{2}\sqrt{r^2-x^2}$

mass $= \sqrt{r^2-x^2}\,dx\cdot\rho$

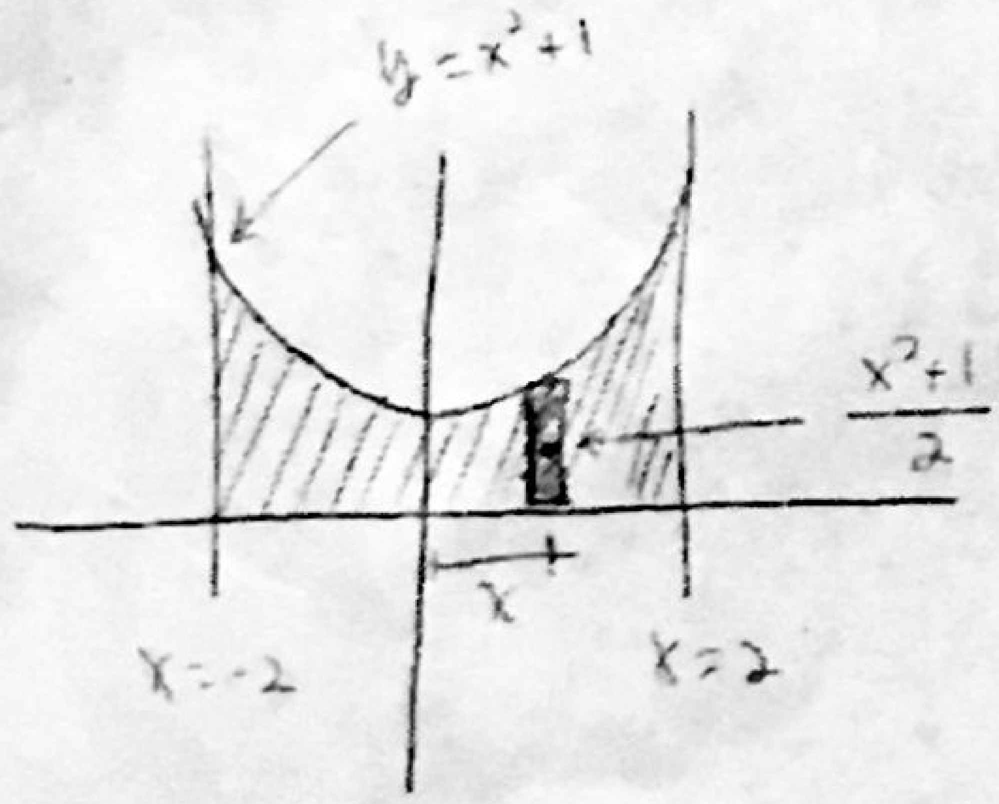
$= \sqrt{r^2-x^2}\,dx$

$$\bar{Y} = \frac{\int_{-r}^{r}\frac{1}{2}\sqrt{r^2-x^2}\cdot\sqrt{r^2-x^2}\,dx}{\int_{-r}^{r}\sqrt{r^2-x^2}\,dx} = \frac{\int_{-r}^{r}\frac{1}{2}(r^2-x^2)\,dx}{\frac{\pi r^2}{2}}$$

$$= \frac{\left[\frac{1}{2}r^2x - \frac{1}{6}x^3\right]_{-r}^{r}}{\frac{\pi r^2}{2}} = \frac{\left(\frac{1}{2}r^3-\frac{1}{6}r^3\right)-\left(-\frac{1}{2}r^3+\frac{1}{6}r^3\right)}{\frac{\pi r^2}{2}} = \frac{\frac{2}{3}r^3}{\frac{\pi r^2}{2}} = \frac{4r^3}{3\pi r^2}$$

COM: $\left(0, \frac{4}{3\pi}r\right)$

4. Find the center of mass of the region between the x-axis and the parabola $y = x^2+1$ between $x=-2$ and $x=2$.



$$\bar{X} = \frac{\int_{-2}^{2} x(x^2+1)\,dx}{\int_{-2}^{2}(x^2+1)\,dx} = 0$$

by symmetry of the region (or can integrate to double check)

$$\bar{Y} = \frac{\int_{-2}^{2}\frac{x^2+1}{2}\cdot(x^2+1)\,dx}{\int_{-2}^{2}(x^2+1)\,dx}$$

$$= \frac{\frac{1}{2}\int_{-2}^{2} x^4+2x^2+1\,dx}{\left[\frac{1}{3}x^3+x\right]_{-2}^{2}}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{5}x^5+\frac{2}{3}x^3+x\right)_{-2}^{2}}{\frac{28}{3}}$$

$$= \frac{\frac{206}{15}}{\frac{28}{3}} = \frac{103}{70} \approx 1.471$$

x^2+1

dx

center $(\tilde{x},\tilde{y})$

$\tilde{x} = x$

$\tilde{y} = \frac{x^2+1}{2}$

mass $= (x^2+1)\,dx\cdot\rho$

$= (x^2+1)\,dx$

Assume $\rho=1$ for similar reasons to #3

COM: $(0, 1.471)$

Notice that this point is outside the region. This sometimes happens when the region has a non-convex shape.