Polar Coordinates (Appendices H1 and H2)

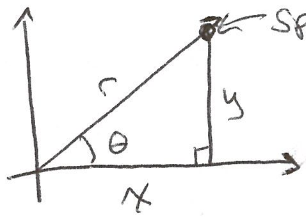
Thanks to Faan Tone Liu and Noah Williams

Key Points:

- Every location in the plane can be described by (r, θ), where

$$r = \text{distance from the origin}$$
$$\theta = \text{angle from the positive } x\text{-axis.}$$

- Consider the following picture:



Specify by (x, y) or (r, θ)

- Converting from polar to rectangular coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

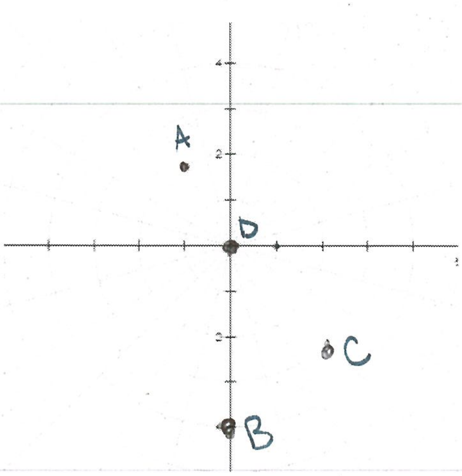
- Converting from rectangular to polar coordinates:

$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$

Examples:

1. Plot the following points:

   A. $(r, \theta) = (2, \frac{2\pi}{3})$
   B. $(r, \theta) = (4, \frac{3\pi}{2})$
   C. $(r, \theta) = (-3, \frac{3\pi}{4})$
   D. $(r, \theta) = (0, \frac{11\pi}{6})$



2. Convert $(2, \frac{2\pi}{3})$ into rectangular coordinates.

$r \nearrow \quad \uparrow \theta$

$x = r\cos(\theta) = 2 \cdot \frac{-1}{2} = -1$

$y = r\sin(\theta) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$\rightarrow \boxed{(-1, \sqrt{3})}$

3. Convert $(-5, -5\sqrt{3})$ into polar coords.

$\nearrow r\cos\theta \quad \uparrow r\sin\theta$

$x^2+y^2=r^2 \rightarrow 25+75=r^2 \rightarrow r = \pm 10$ (use $r=10$)

in QIII $\cos(\theta) = \frac{-1}{2} \rightarrow \theta = \frac{4\pi}{3}$

4. Convert $r = 2$ to rectangular coords.

$x^2+y^2=r^2 \longrightarrow \boxed{x^2+y^2=4}$

5. Convert $r = 3\cos\theta$ to rectangular coords.

$$\begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2=r^2 \end{bmatrix}$$

$\frac{x}{r} = \cos\theta \rightarrow \frac{x}{r} = \frac{r}{3}$

$3x = r^2$

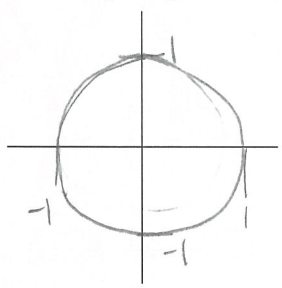
$\rightarrow x^2+y^2 = 3x$

complete □

$\boxed{(x-\frac{3}{2})^2 + y^2 = \frac{9}{4}}$

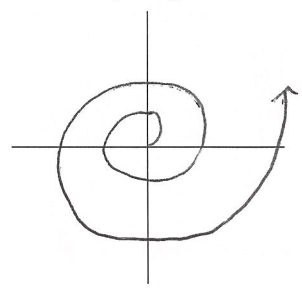
6. Graph the following polar curves:

$r = 1$

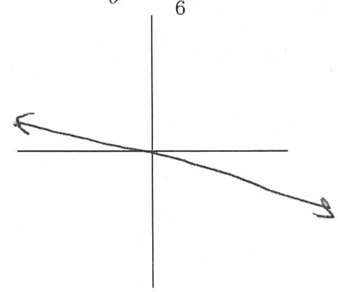


unit circle!

$r = \theta$

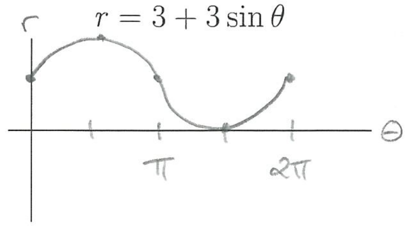


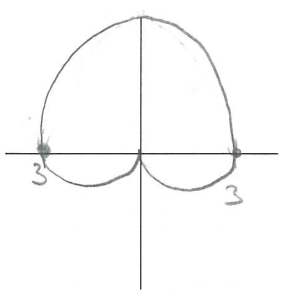
$\theta = \frac{5\pi}{6}$



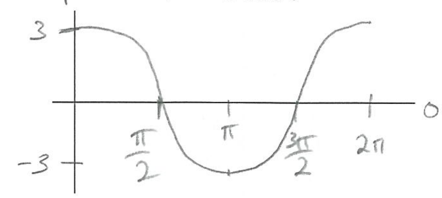
7. Graph the following polar curves (Hint: first graph in rect. coords):

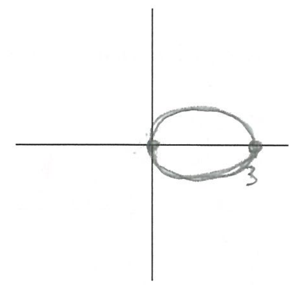
$r = 3 + 3\sin\theta$





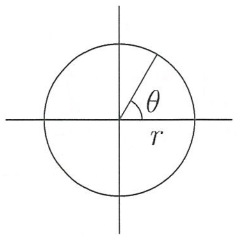
$r = 3\cos\theta$





Key Points:

- Area of a sector:

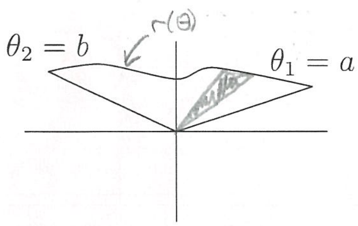


Area of entire circle $= \pi r^2$

Fraction of circle $= \dfrac{\theta}{2\pi}$

Area of sector $= \dfrac{\theta}{2\pi} \cdot \pi r^2 = \boxed{\dfrac{\theta}{2} r^2}$

- Area of a polar region:



Area of thin slice $= \dfrac{\Delta\theta}{2}[r(\theta)]^2$

Estimate of total area $= \frac{1}{2}\sum [r(\theta)]^2 \Delta\theta$

Exact area $= \frac{1}{2}\int_a^b [r(\theta)]^2 \, d\theta$

- To find the slopes of tangent lines to polar curves and arc length of polar curves, use parametric equations:

$x = r\cos\theta = r(\theta) \cdot \cos(\theta)$ ★ r is (usually) changing with respect to θ.

$y = r\sin\theta = r(\theta) \cdot \sin(\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r(\theta)\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{-r(\theta)\sin\theta + \frac{dr}{d\theta}\cos(\theta)}$$

Arc length $= \int_a^b \sqrt{\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2} \, d\theta$

Arc length (simplified) $= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

- Other Notes:

Examples:

1. Find the area inside the region bounded by $r = 3 + 3\sin\theta$

$$\frac{1}{2}\int_0^{2\pi}(3+3\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(9+18\sin(\theta)+9\sin^2(\theta))\,d\theta$$

$$= \frac{9}{2}\int_0^{2\pi}1\,d\theta + 9\int_0^{2\pi}\sin(\theta)\,d\theta + \frac{9}{2}\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta$$

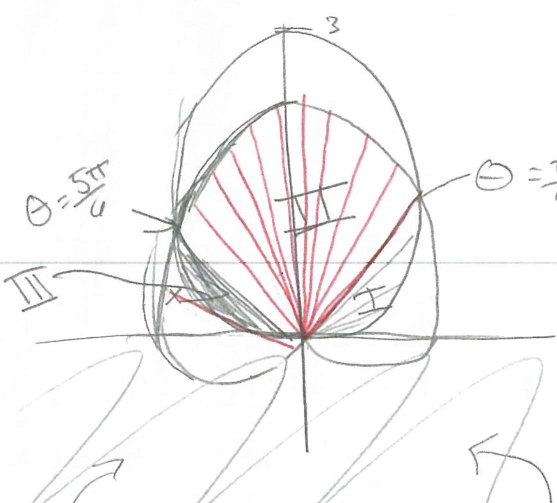
$$= \frac{9}{2}\cdot(2\pi-0) + 9\left(-\cos(\theta)\Big|_0^{2\pi}\right) + \frac{9}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}\right)$$

$$= 9\pi + 9(0) + \frac{9}{4}(2\pi)$$

$$= 9\pi + \frac{9\pi}{2} = \boxed{\frac{27\pi}{2}}$$

2. Find the area of the region that lies inside both $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

Careful sketch



No overlap in these quadrants!

Only need to integrate from $\theta = 0$ to π

Find intersection: $1+\sin(\theta) = 3\sin(\theta)$

$\frac{1}{2} = \sin\theta \to \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\underbrace{\frac{1}{2}\int_0^{\pi/6}(3\sin\theta)^2\,d\theta}_{I} + \underbrace{\frac{1}{2}\int_{\pi/6}^{5\pi/6}(1+\sin\theta)^2\,d\theta}_{II} + \underbrace{\frac{1}{2}\int_{5\pi/6}^{\pi}(3\sin\theta)^2\,d\theta}_{III}$$

Three sections!

$$\frac{9}{2}\int_0^{\pi/6}\sin^2\theta\,d\theta + \frac{1}{2}\int_{\pi/6}^{5\pi/6}(1+2\sin\theta+\sin^2\theta)\,d\theta + \frac{9}{2}\int_{5\pi/6}^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{9}{4}\int_0^{\pi/6}(1-\cos(2\theta))\,d\theta + \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(1+2\sin\theta+\frac{1}{2}(1-\cos2\theta)\right)d\theta + \frac{9}{4}\int_{5\pi/6}^{\pi}(1-\cos2\theta)\,d\theta$$

$$\ldots = \frac{5\pi}{4}$$

3. Find the length of $r = 2\csc\theta$ from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$. What is the slope of the curve at $x = \frac{\pi}{2}$?

$$x = r(\theta)\cdot\cos(\theta) \rightarrow x(\theta) = 2\cdot\frac{1}{\sin(\theta)}\cos(\theta) = 2\cot(\theta)$$

$$\rightarrow x'(\theta) = -2\csc^2(\theta)$$

$$y = r(\theta)\cdot\sin(\theta) \rightarrow y(\theta) = 2$$

$$\rightarrow y'(\theta) = 0$$

Arc length:

$$\int_{\pi/6}^{\pi/2}\sqrt{0^2 + (-2\csc^2\theta)^2}\,d\theta = 2\int_{\pi/6}^{\pi/2}\csc^2(\theta)\,d\theta = -2\cot(\theta)\Big|_{\pi/6}^{\pi/2}$$

$$= 0 - (-2\sqrt{3}) = \boxed{2\sqrt{3}}$$

Slope

$$\frac{dy}{dx} = \frac{0}{-2\csc^2(\theta)} = \boxed{0}$$

4. Find the arc length of the cardioid $3 + 3\sin\theta$.

$$r = 3 + 3\sin\theta$$
$$r'(\theta) = 3\cos\theta$$

$$x = r\cos(\theta) = 3\cos(\theta) + 3\sin\theta\cos\theta$$

$$\rightarrow x'(\theta) = -3\sin(\theta) - 3\sin^2\theta + 3\cos^2\theta$$

$$y = r\sin\theta = 3\sin\theta + 3\sin^2\theta$$

$$\rightarrow y'(\theta) = 3\cos\theta + 6\sin\theta\cos\theta$$

Arc length $= \int_0^{2\pi}\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\,d\theta$ ... gets ugly!

Try other formula:

$$\text{Arc length} = \int_0^{2\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi}\sqrt{9 + 18\sin\theta + 9\sin^2\theta + 9\cos^2\theta}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{18 + 18\sin\theta}\,d\theta = \sqrt{18}\int_0^{2\pi}\sqrt{1+\sin\theta}\,d\theta = \sqrt{18}\int_0^{2\pi}\frac{\sqrt{1+\sin\theta}\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\,d\theta$$

$$= \sqrt{18}\int_0^{2\pi}\frac{|\cos\theta|}{\sqrt{1-\sin\theta}}\,d\theta = \sqrt{18}\left(\int_0^{\pi/2}\frac{\cos\theta\,d\theta}{\sqrt{1-\sin\theta}} - \int_{\pi/2}^{3\pi/2}\frac{\cos\theta\,d\theta}{\sqrt{1-\sin\theta}} + \int_{3\pi/2}^{2\pi}\frac{\cos\theta\,d\theta}{\sqrt{1-\sin\theta}}\right)$$

let $u = 1 - \sin\theta$
$-du = \cos\theta\,d\theta$
change bounds..

$$= \sqrt{18}\left(-\int_1^0 u^{-1/2}\,du + \int_0^2 u^{-1/2}\,du - \int_2^1 6u^{-1/2}\,du\right) = \sqrt{18}\left(-2u^{1/2}\Big|_1^0 + 2u^{1/2}\Big|_0^2 - 2u^{1/2}\Big|_2^1\right)$$

$$= \sqrt{18}(-2(0) + 2 + 2\sqrt{2} - 0 - 2 + 2\sqrt{2}) = \sqrt{18}\cdot 4\sqrt{2} = 24$$