Math 2300: Calculus	**Spring 2019**

Lecture 66: Tuesday April 23

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WebAssign Due Tonight

66.1 Parametric Equations Worksheet

1. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following sets of parametric equations:

(a)

$$x(t) = 4t^3 - t^2 + 7t$$
$$y(t) = t^4 - 6$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

$$\boxed{\frac{dy}{dx} = \frac{4t^3}{12t^2-2t+7}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)}$$

$$= \frac{\frac{d}{dt}\left(\frac{4t^3}{12t^2-2t+7}\right)}{12t^2-2t+7}$$

$$= \frac{\frac{(12t^2-2t+7)(12t^2)-(4t^3)(24t-2)}{(12t^2-2t+7)^2}}{12t^2-2t+7}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{(12t^2-2t+7)(12t^2)-(4t^3)(24t-2)}{(12t^2-2t+7)^3}}$$

(b)

$$x(t) = e^{-7t} + 2$$
$$y(t) = 6e^{2t} + e^{-3t} - 4t$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

$$\boxed{\frac{dy}{dx} = \frac{12e^{2t} - 3e^{-3t} - 4}{-7e^{-7t}}}$$

simplify to make next part easier:

$$\frac{dy}{dx} = \frac{-12}{7}e^{9t} + \frac{3}{7}e^{4t} + \frac{4}{7}e^{7t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)}$$

$$= \frac{\frac{d}{dt}\left(-\frac{12}{7}e^{9t} + \frac{3}{7}e^{4t} + \frac{4}{7}e^{7t}\right)}{-7e^{-7t}}$$

$$= \frac{\frac{-108}{7}e^{9t} + \frac{12}{7}e^{4t} + 4e^{7t}}{-7e^{-7t}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{108}{49}e^{16t} - \frac{12}{49}e^{11t} - \frac{4}{7}e^{14t}}$$

2. Find the equation of the tangent line(s) to the set of parametric equations

$$x(t) = t^2 - 2t - 11$$

$$y(t) = t(t-4)^3 - 3t^2(t-4)^2 + 7$$

at $(-3, 7)$.

$$\frac{dy}{dx} = \text{slope of tangent line} = \frac{y'(t)}{x'(t)} = \frac{(t-4)^3 + 3t(t-4)^2 - 6t(t-4)^2 - 6t^2(t-4)}{2t-2}$$

At what t is $x=-3$, $y=7$?

$$-3 = t^2 - 2t - 11$$
$$0 = t^2 - 2t - 8$$
$$0 = (t-4)(t+2)$$
$$t=4,\ t=-2.$$

$$y(-2) = -2(-6)^3 - 3(4)(-6)^2 + 7$$

$$y(-2) = 7$$
$$y(4) = 7$$

$\rightarrow$ We will get two tangent lines: one for $t=-2$, one for $t=4$.

slope @ $t=-2$:

$$= \frac{(-2-4)^3 + 3(-2)(-2-4)^2 - 6(-2)(-2-4)^2 - 6(-2)^2(-2}{2(-2)-2}$$

$$= -24$$

$\rightarrow$ Tangent line: $\boxed{y = -24(x+3)+7}$

slope @ $t=4$:

$$=0$$

$\rightarrow$ Tangent line: $\boxed{y=7}$

(horizontal line)

3. Determine the length of the parametric curve given by the set of parametric equations:

$$x(t) = 8t^{3/2} \quad \rightarrow \quad x'(t) = 12\sqrt{t}$$

$$y(t) = 3 + (8-t)^{3/2} \qquad y'(t) = -\frac{3}{2}\sqrt{8-t}$$

$$0 \le t \le 4.$$

$$\text{Arc Length} = \int_0^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^4 \sqrt{144t + \frac{9}{4}(8-t)}\, dt$$

$$= \int_0^4 \sqrt{\frac{576t}{4} - \frac{9t}{4} + 18}\, dt$$

$$= \int_0^4 \sqrt{\frac{567t}{4} + 18}\, dt$$

let $u = \frac{567t}{4} + 18$

$du = \frac{567}{4}\,dt$

$\frac{4}{567}\,du = dt$

Change bounds too: $t=4 \rightarrow u=585$

$t=0 \rightarrow u=18$

$$= \int_{18}^{585} u^{1/2} \cdot \frac{4}{567}\,du$$

$$= \frac{4}{567}\int_{18}^{585} u^{1/2}\,du$$

$$= \frac{4}{567} \cdot \frac{2}{3} u^{3/2} \Big|_{18}^{585}$$

$$= \frac{4}{567} \cdot \frac{2}{3}\left(585^{3/2} - 18^{3/2}\right)$$

$$\approx \boxed{66.186}$$

4. A steady wind blows a kite due west. The kit's height above the ground from horizontal position $x = 0$ to $x = 100$ is given $y = 100 - \frac{1}{40}(x+20)^2$. Find the distance traveled by the kite.
Hint: You can use the following fact:

$$\int \sqrt{1+u^2}du = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2}) + C$$

$$\boxed{\begin{aligned} x(t) &= t \\ y(t) &= 100 - \tfrac{1}{40}(x+20)^2 \\ \to x'(t) &= 1 \\ y'(t) &= -\tfrac{1}{20}(x+20) \end{aligned}}$$

$$\text{Arc length} = \int_0^{100} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{100} \sqrt{1 + \left(\tfrac{-1}{20}(x+20)\right)^2}\, dt$$

let $u = \frac{-1}{20}(x+20) \to du = -\frac{1}{20}dx$
$\to -20\,du = dx$

when $x=0$, $u=-1$
when $x=100$, $u=-6$

$$= -20\int_{-1}^{-6} \sqrt{1+u^2}\, du = 20\int_{-6}^{-1} \sqrt{1+u^2}\, du$$

use formula given:

$$\text{Arc length} = 20\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln(u+\sqrt{1+u^2})\Big|_{-6}^{-1}\right] = 20(18.3464) = \boxed{366.9}$$

5. A hawk flying at 11 m/s at an altitude of 100 m accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation $y = 100 - \frac{x^2}{33}$ until it hits the ground, where y is its height above the ground and x ist he horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground. Express your answer correct to the nearest tenth of a meter.
Hint: You may want to use the hint from the previous problem.

$$\boxed{\begin{aligned} x(t) &= t \\ y(t) &= 100 - \tfrac{x^2}{33} \\ x'(t) &= 1 \\ y'(t) &= -\tfrac{2}{33}x \end{aligned}}$$

Starts @ $t=0$. Hits ground when $y(t) = 0$:

$100 = \frac{t^2}{33}$

$3300 = t^2 \to t = \pm 57.4$

$\to$ Choose positive t b/c hits ground after dropped @ $t=0$.

$$\text{Arc length} = \int_0^{57.4} \sqrt{1+\left(\tfrac{-2}{33}x\right)^2}\, dx$$

let $u = \frac{-2}{33}x \to du = \frac{-2}{33}dx \to \frac{-33}{2}du = dx$

When $x=0$, $u=0$
when $x = 57.4$, $u = -3.5$

$$= -\frac{33}{2}\int_0^{-3.5} \sqrt{1+u^2}\, du = \frac{33}{2}\int_{-3.5}^{0} \sqrt{1+u^2}\, du$$

use formula:

$$\text{Arc length} = \frac{33}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln(u+\sqrt{1+u^2})\Big|_{-3.5}^{0}\right]$$

$$= \frac{33}{2}(7.35296) = \boxed{121.3}$$

6. Find the average value of the function

$$f(x) = \cos^5(x)\sin^3(x)$$

on the interval $[0, \pi/2]$. Find $c \in [0, \pi/2]$ such that $f(c)$ is equal to the average value. (When finding c, you can use a calculator or wolfram alpha, etc.)

$$f_{ave} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos^5(x)\underbrace{\sin^3(x)}_{=\sin^2 x \cdot \sin x}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos^5(x)(1-\cos^2(x))\sin(x)\,dx$$

let $u = \cos(x)$ $\quad du = -\sin(x)\,dx$

when $x = 0$, $u = 1$

when $x = \pi/2$, $u = 0$

$$= -\frac{2}{\pi}\int_1^0 u^5(1-u^2)\,du$$

$$= \frac{2}{\pi}\int_0^1 (u^5 - u^7)\,du$$

$$= \frac{2}{\pi}\left(\frac{u^6}{6} - \frac{u^8}{8}\Big|_0^1\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{6} - \frac{1}{8}\right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{24} = \boxed{\frac{1}{12\pi}}$$