Math 2300: Calculus **Spring 2019**

Lecture 65: Monday April 22

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65.1 Warm-Up

A ray is parametrized by

$$x(t) = 2 + 3t$$

$$y(t) = -1 + 5t$$

where $t \geq 0$. *Note that we are imposing this restriction on t!
If we hadn't said anything, we would consider $t \in (-\infty, \infty)$.

(a) Does $(5, 4)$ lie on the ray?

(b) Does $(2, 1)$ lie on the ray?

(c) Does $(-1, -6)$ lie on the ray?

(d) When does the line hit the y-axis?

(e) What is the speed of the motion along the line?

(f) What is the slope of the line?

Solution:

(a) $x = 5$ when $t = 1$. When $t = 1$, $y = 4$, so YES: the point $(5, 4)$ does lie on the ray.

(b) $x = 2$ when $t = 0$. When $t = 0$, $y = -1$, so NO: the point $(2, 1)$ does not lie on the ray.

(c) $x = -1$ when $t = -1$, and the ray only considers $t \geq 0$, so the point does NOT lie on the ray.

(d) The y-intercept occurs when $x = 0$. $x = 0$ when $t = -2/3$. Since the ray is only for $t \geq 0$, the ray does not hit the y-axis.

(e) We recall the equation for speed:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Since $x'(t) = 3$ and $y'(t) = 5$, the speed in this case is $\sqrt{9 + 25} = \sqrt{34}$.

(f) We recall the formula for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{5}{3}.$$

65.2 The Calculus of Parametric Equations, Part II

Recall:

- We are thinking of parametric equations as describing the path taken by a snail.
- We have talked about describing this path in two ways: parametric and cartesian equations.
- We have talked about the various speeds associated to the path ($\frac{dx}{dt}, \frac{dy}{dt}, \frac{dy}{dx}$.

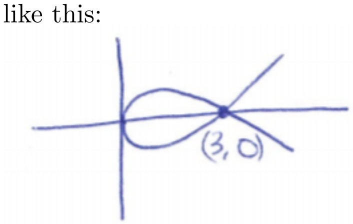
Something else we might be interested in:

- Calculating a **second derivative** $\frac{d^2y}{dx^2}$
- Calculating how far our snail has travelled: This is called **arc length**.
- Calculating the **average speed** of a snail along a path.

65.2.1 Second Derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}$$

The first equality is chain rule: We temporarily think of t as a function of x. The second equality is just simplifying.
This equation still tells us about the concavity of the path. This is useful, because some of the paths cross like this:



so it's useful to talk about the concavity at a *time* instead of at an x-value.
Let's put it to use!

65.2.1.1 Example

Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.

$$x(t) = 1 - t^2$$
$$y(t) = t^7 + t^5$$

Solution: Using the equation for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{7t^6+5t^4}{-2t}\right)}{-2t} = \frac{\frac{d}{dt}(\frac{-7}{2}t^5 + \frac{-5}{2}t^3)}{-2t} = \frac{\frac{-35}{2}t^4 + \frac{-15}{2}t^2}{-2t} = \frac{1}{4}t(35t^2 + 15)$$

This equation is equal to 0 when $t = 0$.
When $t > 0$, we can plug in a test value and we find that $\frac{d^2y}{dx^2} > 0$, so the curve is concave up.
When $t < 0$, we can plug in a test value and we find that $\frac{d^2y}{dx^2} < 0$, so the curve is concave down.

65.2.2 Area Enclosed by Parametric Curves

The area under the curve $y = F(x)$ from a to b is given

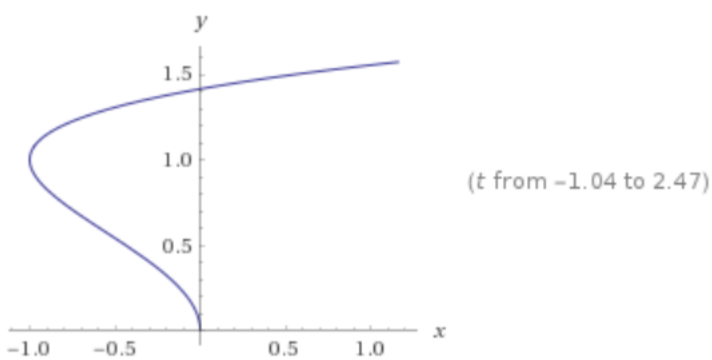
$$A = \int_a^b F(x)dx.$$

If the curve is traced out by the parametric equations $x(t)$ and $y(t)$, for $\alpha \leq t \leq \beta$, then we can calculate the area:

$$A = \int_a^b y dx = \int_\alpha^\beta y(t)x'(t)dt.$$

65.2.2.1 Example

Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$, and the y-axis.



This curve hits the y-axis when $x = 0$, so at times $t = 0, t = 2$. Note that t cannot be negative, since $y = \sqrt{t}$. Using the formula above:

$$\begin{aligned} A &= \int_0^2 \sqrt{t}(2t-2)dt \\ &= \int_0^2 (2t^{3/2} - 2t^{1/2})dt \\ &= \frac{2t^{5/2}}{5/2} - \frac{2t^{3/2}}{3/2}dt|_0^2 \\ &= \frac{4\sqrt{2^5}}{5} - \frac{4\sqrt{2^3}}{3} \end{aligned}$$

65.2.3 Average Speed

This one should be familiar. We know that the function that gives us the speed is:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

We recall from Calculus 1 that the average value of a function on $[a, b]$ is given by:

$$\frac{1}{b-a}\int_a^b f(x)dx,$$

so **average speed** is just a special case of this. The average speed on a parametric function is:

$$\frac{1}{b-a}\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

65.2.3.1 Example

Find the average speed of the snail whose path is given

$$x(t) = 3\sin(t)$$

$$y(t) = 3\cos(t)$$

from $t = 0$ to $t = \pi$.
Solution:

$$\begin{aligned}
\text{Average Speed} &= \frac{1}{\pi - 0}\int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \frac{1}{\pi}\int_0^\pi \sqrt{(3\cos(t))^2 + (-3\sin(t))^2}\,dt \\
&= \frac{3}{\pi}\int_0^\pi \sqrt{\cos^2(t) + \sin^2(t)}\,dt \\
&= \frac{3}{\pi}\int_0^\pi 1dt \\
&= 3
\end{aligned}$$

65.2.4 Arc Length

Arc length is the integral of speed: The length of the arc between $t = a$ and $t = b$ on the path whose parametric equations is given by $x(t)$ and $y(t)$ is given:

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

65.2.4.1 Example

Determine the length of the parametric curve given by the following parametric equations:

$$x(t) = 3\sin(t)$$

$$y(t) = 3\cos(t)$$

$$0 \le t < 2\pi$$

Solution:
This will be fun, because we will be able to check the answer we get from memorizing the formula for the circumferences of a circle! This is a circle radius 3, so the "arc length" of the whole thing should be $2\pi r = 6\pi$. Let's see:

$$\begin{aligned}
\text{Arc Length } &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^{2\pi} \sqrt{(3\cos(t))^2 + (-3\sin(t))^2} dt \\
&= 3\int_0^{2\pi} \sqrt{\cos^2(t) + \sin^2(t)} dt \\
&= 3\int_0^{2\pi} dt \\
&= 6\pi
\end{aligned}$$