Math 2300: Calculus **Spring 2019**

Lecture 64: Friday April 19

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WebAssign due tonight.

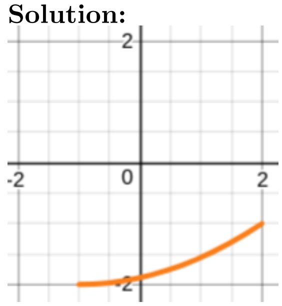
64.1 Warm-Up

Consider the parametric equations:

$$x(t) = 3t - 1$$

$$y(t) = t^2 - 2$$

(a) By plugging in values for t, sketch the shape of the curve for $t \in [0, 1]$.
Solution:



(b) Eliminate the parameter to find a Cartesian equation of the curve.
Solution:
It's easy to solve for t in the x-equation. Then we can plug into the y-equation:

$$\frac{1}{3}(x+1) = t$$
$$y = t^2 - 2$$
$$y = \frac{1}{9}(x+1)^2 - 2$$

(c) Does this shape make sense with what you sketched?
Solution:
Hopefully! You should have sketched the base of a wide-based parabola that opens face up.

64.2 The Calculus of Parametric Equations

Now that we have parametric equations, we want to translate the language of calculus so that we can use this new tool.
What would derivatives even correspond to?
Let's start with parametric equations $x(t), y(t)$. Think about them describing *motion* in the xy plane: A snail is moving along your paper with equations described by $x(t)$, $y(t)$.
Then:

1. $$\frac{dx}{dt} = \text{ the instantaneous velocity in the } x \text{ direction}$$

2. $$\frac{dy}{dt} = \text{ the instantaneous velocity in the } y \text{ direction}$$

3. $$\frac{dy}{dx} = \text{ the rate of change in } y \text{ with respect to } x\text{; the slope of the tangent line}$$

If we have parametric equations, it's pretty straight-forward to get the $\frac{dx}{dt}, \frac{dy}{dt}$ derivatives: Differentiate each parametric equation with respect to the parameter t.
How do we get $\frac{dy}{dx}$? Assuming we don't want to go through all the work of converting to a Cartesian equation, we can use the parametric equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

So we just need to divide the derivatives. Notice that this will not eliminate the parameter t. The advantage to this is that we can still think of this equation as describing motion over time.
SO FAR: We can describe the velocity of the snail in the x-direction. We can describe the velocity of the snail in the y-direction. What about the overall speed of the snail?
The **instantaneous speed** of the snail along the curve as a function of t is given:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

This follows from the Pythagorean theorem! Think about why.

64.2.1 Examples

Consider the parametric equations:

$$x(t) = t^3 - 4$$

$$y(t) = 4t - t^2$$

1. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$$t = 1$$

Solution:
In order to write a tangent line equation, we need two things: A point and a slope.
To find the point, plug in $t = 1$ to the parametric equations:

$$x(1) = 1 - 4 = -3$$

$$y(1) = 4 - 1 = 3$$

So the point of tangency is $(x, y) = (-3, 3)$. This occurs at t (time) $= 1$.
Now, find the slope of this tangent line at $t = 1$:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{4-2t}{3t^2}$$

Evaluating at $t = 1$:

$$\text{Slope} = \frac{4-2}{3} = \frac{2}{3}.$$

Putting the point and the slope together, we get the equation of the tangent line:

$$y = \frac{2}{3}(x+3) + 3$$

2. Find the point(s) on the curve where the tangent is horizontal.
Solution:
We are looking for where $\frac{dy}{dx} = 0$:

$$\begin{aligned} 0 &= \frac{dy}{dx} \\ 0 &= \frac{y'(t)}{x'(t)} \\ 0 &= \frac{4-2t}{3t^2} \\ 0 &= 4 - 2t \\ t &= 2 \end{aligned}$$

So the curve has a horizontal tangent line when $t = 2$. To find out what point this corresponds to, plug in $t = 2$ into the parametric equations:

$$x(2) = 8 - 4 = 4$$

$$y(2) = 8 - 4 = 4$$

So the point $(4, 4)$ has a horizontal tangent line.

3. Graph on desmos for an appropriate interval of t-values to check.
Solution: