Math 2300: Calculus **Spring 2019**

Lecture 62: Wednesday April 17

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62.1 Parametric Equations

A **parametric equation** is one in which the coordinates are given by independent equations.

We have seen one example of this already in the unit circle. We can think of the equation of a circle as a regular equation $x^2 + y^2 = 1$,
..OR we can think of the points on the unit circle as being given $(\cos(\theta), \sin(\theta))$.
The x coordinate is given by $\cos(\theta)$, the y coordinate is given by $\sin(\theta)$: these are the parametric equations.
We call θ the **parameter**.
Often, we use t to denote the parameter. (θ is usually just for angles, but the parameters in our parametric equations won't always be angles.)

There are (basically) three things we want to be able to do with parametric equations:

- Convert from parametric equations to a cartesian equation (regular "$y =$" style)
- Sketch the curves given by parametric equations.
- Write a parametric equation that describes a certain path.

62.1.1 Converting to Cartestian

The process for this is solving each of the parametric equations for t and then setting them equal to each other.

62.1.1.1 Simple Example

$$x(t) = 2t + 2$$
$$y(t) = t - 1$$

To get this parametric equation in cartesian form, solve each equation for t:

$$\frac{1}{2}(x - 2) = t$$
$$y + 1 = t$$

Then set the t equations equal to each other and solve for y:

$$y + 1 = \frac{1}{2}(x - 2)$$
$$y = \frac{1}{2}x - 2$$

Done! This was like solving a system of equations by elimination.

62.1.1.2 Trickier Example

$$x(t) = t^2 + t,\ y(t) = 2t - 1$$

Again, begin by solving for t.
This time it's easy to solve for t in terms of y:

$$\frac{1}{2}(y+1) = t$$

Then we can just plug this into the $x(t)$ equation.
This will be like solving a system of equations with substitution.

$$x = (\frac{1}{2}(y+1))^2 + \frac{1}{2}(y+1)$$
$$x = \frac{1}{4}(y^2 + 2y + 1) + \frac{1}{2}y + \frac{1}{2}$$
$$x = \frac{1}{4}y^2 + y + \frac{3}{4}$$

This equation is a parabola.

62.1.2 Sketching

Consider the parametric equation

$$x(t) = 5\cos(t)$$

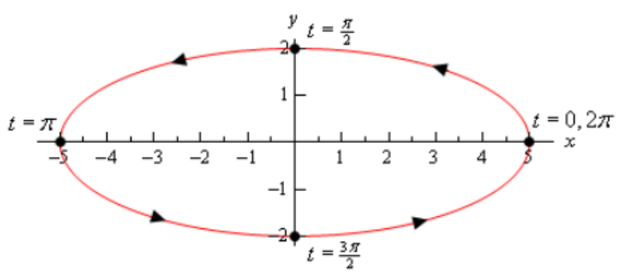
$$y(t) = 2\sin(t)$$

What do you expect it to look like?
It's kind of like the unit circle parametric equation:

$$x(t) = \cos(t)$$

$$y(t) = \sin(t)$$

But the x and y's are stretched. So Yes! It's an ellipse:



Notice the arrows indicating the direction. Think about where it starts at $t = 0$, and what happens as t increases.
This is the type of thing you should be able to sketch without switching to Cartesian.
BUT if you needed to switch this to Cartesian, use the fact that

$$\cos^2(t) + \sin^2(t) = 1:$$

$$\frac{x}{5} = \cos(t)$$
$$\frac{y}{2} = \sin(t)$$
$$\Rightarrow$$
$$\cos^2(t) + \sin^2(t) = 1$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

62.1.3 Writing a parametric equation

Find parametric equations for a line segment from $(-2, 8)$ to $(1, 2)$.
Solution:
Sketch what this line segment looks like. We'll make parametric equations for x and y for a given parameter that goes from 0 to 1 (we choose 0 to 1 because it's easy and makes sense: starts at 0, ends at 1).
Both the x and y equations will be lines:

$$x(t) = m_1 t + b_1$$
$$y(t) = m_2 t + b_2$$

We need to solve for m_1, m_2, b_1, b_2.
When $t = 0$, $x = -2$ and when $t = 1$, $x = 1$. Plug this in and solve for m_1, b_1:

$$\begin{aligned} -2 &= m_1(0) + b_1 \\ -2 &= b_1 \\ 1 &= m_1(1) - 2 \\ 3 &= m_1 \end{aligned}$$

So the x-equation is:

$$x(t) = 3t - 2$$

Do the same thing for the y equation:

$$\begin{aligned} 8 &= m_2(0) + b_2 \\ 8 &= b_2 \\ 2 &= m_1(1) + 8 \\ -6 &= m_1 \end{aligned}$$

So the y parametric equation is:

$$y(t) = -6t + 8$$

So our parametric equations are:

$$x(t) = 3t - 2$$
$$y(t) = -6t + 8$$

with parameter $t \in [0, 1]$.

Soon, we will learn an alternative way of solving this, with calculus!