Math 2300: Calculus **Spring 2019**

Lecture 61: Tuesday April 16

Lecturer: Sarah Arpin

ebAssign due tonight.

61.1 The Logistic Equation

We have already seen the differential equation that models exponential growth:

$$\frac{dP}{dt} = kP$$

but this model is not useful for long-term population modeling, because it results in infinite growth eventually. We know this is not possible.

More realistic models for population growth take environmental factors into account: every population has a **carrying capacity** based on the population's survival needs and the environment they inhabit.

Let M denote this constant carrying capacity for the population P. The growth of population P can be modeled by the differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where k is a positive constant.

Notice some features of this differential equation:

- When $P > M$, $\frac{dP}{dt}$ is negative. In real life, this means that when the population is over the carrying capacity, the population starts to decline.
- When $P = M$, $\frac{dP}{dt} = 0$. In real life, this means that when the population is equal to its carrying capacity, it neither increases nor decreases.
- When $P < M$, $\frac{dP}{dt}$ is negative. In real life, this means that when the population is below carrying capacity, it grows.

61.1.1 Differential Equation to Solution

Let's start with the logistic growth differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

and an initial condition:

$$P(0) = P_0.$$

This is a little tricky to solve (you should do it yourself as practice - there's a partial fractions integral!), but we can check that the equation:

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0}.$$

To check, first check the initial condition:

$$P(0) = \frac{M}{1 + Ae^0} = \frac{M}{1 + \frac{M-P_0}{P_p}} = \frac{MP_0}{P_0 + (M - P_0)} = P_0$$

Now, check the derivative:

$$\frac{dP}{dt} = M(1 + Ae^{-kt})^{-2}Ake^{-kt}$$

Look for and try to plug in $P(t) = \frac{M}{1 + Ae^{-kt}}$:

$$= k \cdot \frac{M}{1 + Ae^{-kt}} \cdot \frac{Ae^{-kt}}{1 + Ae^{-kt}}$$

$$= kP \cdot \frac{Ae^{-kt}}{1 + Ae^{-kt}}$$

Here's a slick "add-one-subtract-one" trick:

$$= kP \cdot \frac{1 + Ae^{-kt} - 1}{1 + Ae^{-kt}}$$

$$= kP \cdot \left(\frac{Ae^{-kt} + 1}{1 + Ae^{-kt}} - \frac{1}{1 + Ae^{-kt}}\right)$$

$$= kP\left(1 - \frac{P}{M}\right)$$

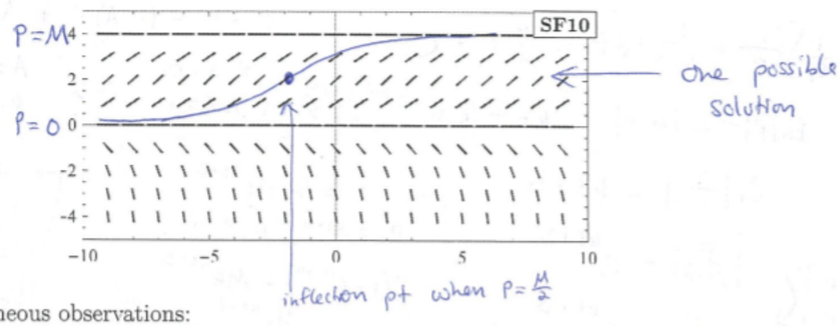
Three points:

- You should be able to derive this by using separation of vairables.
- You should be able to do the above check to show that the equation is indeed the solution.
- ...But if you are given a question and not specifically asked to do either of those two things, you can just memorize the fact that $P(t)$ is the solution to the logistic growth differential equation.

61.1.2 Slope Fields

(Borrowed from Noah Williams's notes)

- The slope field for the logistic growth equation is



- Miscellaneous observations:
  - If P is small, then $\frac{dP}{dt} \approx$ kP (basically exponential growth).
  - If $P \approx M$, then $\frac{dP}{dt} \approx$ 0 (growth slows to 0).
  - Equilibrium (constant) solutions are: $P=0$, $P=M$.
  - If the population starts between 0 and M: $\lim_{t\to\infty} P(t) =$ M.
  - Using Calc I methods, we can show that $P(t)$ has an inflection point when $P=\frac{M}{2}$.

61.1.3 Example

The population of the US in 1800 and 1850 was 5.3 and 23.1 million people, respectively.

1. Predict its population in 1900 and in 1950 using the exponential model for population growth.
2. In 1900, the population of the US was actually only 76 million people. Using this fact, create a logistic model of population growth.
3. Use the logistic model from (b) to correct your prediction about the population in 1950.

Solution:

1. The differential equation for exponential growth is

$$\frac{dP}{dt} = kP,\ P(0) = P_0.$$

   The solution of this equation is:

$$P(t) = P_0 e^{kt}.$$

   In our particular example, we have $P(0) = 5.3$ million people, with $t = 0$ corresponding to the year 1800. This gives:

$$P(t) = 5.3e^{kt}$$

We also know $P(50) = 23.1$ million people, and using this we can solve for k:

$$\begin{aligned} 23.1 &= 5.3e^{50k} \\ \frac{23.1}{5.3} &= e^{50k} \\ \ln\left(\frac{23.1}{5.3}\right) &= 50k \\ \frac{1}{50}\ln\left(\frac{23.1}{5.3}\right) &= k \end{aligned}$$

Now that we have k, we have the solution:

$$P(t) = 5.3e^{\frac{1}{50}\ln\left(\frac{23.1}{5.3}\right)t}$$

and we can use this to solve $P(100)$, $P(150)$:

$$P(100) = 100.7 \text{ million people}$$

$$P(150) = 438.8 \text{ million people}$$

These seem unreasonable...since the exponential growth model is unreasonable.

2. If we switch to the logistic growth model, our differential equation is:

$$\frac{dP}{dt} = kP(1 - P/M),\ P(0) = P_0.$$

The solution equation looks like:

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0}$$

Again putting in our initial condition $P_0 = 5.3$ million people:

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - 5.3}{5.3}$$

We need to solve for M and k now, but we have two pieces of information, so we can solve for 2 variables:

We know

$$P(50) = 23.1,\ P(100) = 76$$

Plugging these in:

$$23.1 = \frac{M}{1 + Ae^{-50k}},\ 76 = \frac{M}{1 + Ae^{-100k}}$$

Using Desmos or some other computer algebra system to solve this system of equations, we get:

$$k \approx 0.031476,\ M = 189.4$$

Now we have our solution:

$$P(t) = \frac{189.4}{1 + Ae^{-0.031476t}}, \text{ where } A = \frac{189.4 - 5.3}{5.3} = 34.74$$

3. Let's re-do $P(150)$ with this equation:

$$P(150) = 144.7 \text{ million people}$$

This problem used real data for the population in 1800, 1850, and 1900. The population in 1950 was actually 152.3 million people, which is pretty close to what we got!