Differential Equations: Applications (Ch 7)

Key Points:

- If the rate at which $y(t)$ changes is proportional to the value of y at a time t, we have the differential equation
$$\frac{dy}{dt} = ky.$$
Some examples where this occurs include:

- **Newton's Law of Cooling** states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings (provided that this difference is not too large). This is summarized in the differential equation
$$\frac{dT}{dt} = k(T - T_s),$$
where $T = T(t)$ is the temperature of the object at time t, and T_s is the temperature of the surroundings.

Exercises:

1. Strontium-90 has a half-life of 28 days.

   (a) Write down a differential equation to model this situation. Let $m = m(t)$ be the mass of strontium after t days.

   $$\frac{dm}{dt} = k \cdot m$$

   (b) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.

   $$\frac{dm}{dt} = km$$
   $$\int \frac{1}{m}\,dm = \int k\,dt$$
   $$\ln|m| = kt + C$$
   $$|m| = e^C e^{kt}$$

   Initial Condition

   $$m(0) = 50$$
   $$|50| = e^{k \cdot 0 + C}$$
   $$50 = e^C$$
   $$|m| = 50e^{kt}$$
   $$m = 50e^{kt}$$
   (m always ≥ 0)

   Find k:

   $$m(28) = 25$$
   $$25 = 50e^{k \cdot 28}$$
   $$\tfrac{1}{2} = e^{28k}$$
   $$\ln(\tfrac{1}{2}) = 28k$$
   $$k = \frac{\ln(\frac{1}{2})}{28}$$

   $$\boxed{\begin{aligned} m(t) &= 50e^{\ln(\frac{1}{2})/28\, t} \\ m(t) &= 50\left(\tfrac{1}{2}\right)^{t/28} \end{aligned}}$$

   (c) Find the mass remaining after 40 days.

   $$m(40) = 50\left(\tfrac{1}{2}\right)^{40/28} \approx 28.06 \text{ mg}$$

   (d) How long does it take the sample to decay to a mass of 2mg?

   $$2 = 50\left(\tfrac{1}{2}\right)^{t/28}$$
   $$\frac{1}{25} = \left(\tfrac{1}{2}\right)^{t/28}$$
   $$\ln(\tfrac{1}{25}) = \tfrac{t}{28}\ln(\tfrac{1}{2})$$
   $$t = \frac{28\ln(\frac{1}{25})}{\ln(\frac{1}{2})} \approx \boxed{130 \text{ days}}$$

2. When *E. coli* grows in a nutrient-broth meduim each cell divides into two cells every 20 minutes. Suppose the initial population of a culture of E. coli is 60 cells.

(a) Find the relative growth rate (i.e. k).

Let P(t) = # cells after t minutes:

$$P(t) = P_0 e^{kt}$$

(b) Find an expression for the number of cells after t hours.

Initial Population $P(0) = 60$

$$P(t) = 60 e^{kt}$$

Doubles every 20 min

$$120 = 60 e^{k \cdot 20}$$

$$2 = e^{k \cdot 20}$$

$$\ln(2) = k \cdot 20$$

Relative growth rate (part a) $\rightarrow$ $\boxed{k = \frac{\ln(2)}{20}}$

$$P(t) = 60 e^{\frac{\ln(2)}{20} t}$$

$$P(t) = 60 \left(e^{\ln(2)}\right)^{t/20}$$

$$\boxed{P(t) = 60 \, 2^{t/20}}$$

(c) Find the number of cells after 8 hours.

$$P(8\text{hrs}) = P(8 \cdot 60) = 60 \cdot 2^{480/20} \approx 1 \times 10^9 \text{ cells.}$$

(d) Find the rate of growth after 8 hours.

$$P'(t) = 60 \cdot 2^{t/20} \ln(2) \cdot \tfrac{d}{dt}(t/20) = 60 \ln(2) \, 2^{t/20} \cdot 1/20 = 3\ln(2) 2^{t/20}$$

$$P'(8\text{hrs}) = P'(480) = 3\ln(2) 2^{480/20} \approx 3 \times 10^7 \text{ cells/min.}$$

Could also use fact that $\frac{dP}{dt} = kP$

$$\left.\frac{dP}{dt}\right|_{480} = \frac{\ln(2)}{20} \cdot P(480) \approx 3 \times 10^7 \text{ cells/min}$$

(e) When will the population reach 20,000 cells?

$$20000 = 60 \, 2^{t/20}$$

$$\frac{20000}{60} = 2^{t/20}$$

$$\ln\left(\tfrac{2000}{6}\right) = t/20 \ln(2)$$

$$t = \frac{\ln\left(\frac{2000}{6}\right) \cdot 20}{\ln(2)}$$

$$\approx 167.6 \text{ min}$$

$$\approx \boxed{2.79 \text{ hrs}}$$

3. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

Newton's Law of cooling:

$$\frac{dT}{dt} = k(T - T_s)$$

T = temp at time t
k = const
T_s = temp of surroundings

$$\frac{dT}{dt} = k(T-20)$$

$$\int \frac{1}{T-20}\, dT = \int k\, dt$$

$$\ln|T-20| = kt + C$$

$$|T-20| = e^c e^{kt}$$

Temp of coffee > temp of room

$$T - 20 = e^c e^{kt}$$

Solve for C:

$$T(0) = 95^\circ \text{Celsius}$$

$$95 - 20 = e^c e^{k \cdot 0}$$

$$75 = e^c$$

$$T - 20 = 75 e^{kt}$$

Solve for k:

$$\left.\frac{dT}{dt}\right|_{T=70^\circ} = -1^\circ C/\text{min}$$

$$k(70-20) = -1$$

$$k(50) = -1$$

$$k = -1/50$$

$$T - 20 = 75 e^{-1/50 \cdot t}$$

$$\boxed{T(t) = 75 e^{-1/50\, t} + 20}$$

Want time when $T = 70^\circ C$

$$70 = 75 e^{-1/50\, t} + 20$$

$$\frac{50}{75} = e^{-t/50}$$

$$\ln\left(\frac{50}{75}\right) = -t/50$$

$$\boxed{\begin{gathered} t = -50 \ln\left(\frac{50}{75}\right) \\ \approx 20 \text{ minutes later} \end{gathered}}$$

* key is that $T - T_s$ behaves exponentially, so soln to diff-eq is $T - T_s = (T_0 - T_s)e^{kt}$ looks like $P = P_0 e^{kt}$

4. Experiments show that the reaction $H_2 + Br_2 \rightarrow 2HBr$ satisfies the rate law

$$\frac{d[HBr]}{dt} = k[H_2][Br_2]^{1/2},$$

where [HBr], $[H_2]$, and $[Br_2]$ represent the concentrations of molecules of Hydrogen Bromide, Hydrogen, and Bromine, respectively. Suppose that the initial concentrations of the reactants are $[H_2]_0 = a$ moles/L and $[Br_2]_0 = b$ moles/L.

(a) Write a differential equation that relates the rate of change of the concentration [HBr] to the concentrations $[H_2]$ and $[Br_2]$. (**Hint:** Let $x = x(t)$ be the concentration of [HBr] at time t.)

when x $\frac{\text{moles}}{L}$ of HBr, it means we consumed $\frac{x}{2}$ $\frac{\text{moles}}{L}$ of H_2 and of Br_2

$$\frac{dx}{dt} = k\left(a-\frac{x}{2}\right)\left(b-\frac{x}{2}\right)^{1/2}$$

(b) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$ (Why does this make sense?).

$$\frac{dx}{dt} = k\left(a-\frac{x}{2}\right)\left(a-\frac{x}{2}\right)^{1/2} = k\left(a-\frac{x}{2}\right)^{3/2}$$

$$\int \left(a-\frac{x}{2}\right)^{-3/2} dx = \int k\, dt$$

$u = a - \frac{x}{2}$

$du = -\frac{1}{2}dx$

$-2du = dx$

$$-2\int u^{-3/2}\, du = \int k\, dt$$

$$4u^{-1/2} = kt + C$$

$$4\left(a-\frac{x}{2}\right)^{-1/2} = kt + C$$

Solve for C!

$x(0) = 0$ because initially, we only have H_2 & Br_2 and no HBr.

$$4(a)^{-1/2} = 0 + C$$

$$C = \frac{4}{\sqrt{a}}$$

$$4\left(a-\frac{x}{2}\right)^{-1/2} = kt + \frac{4}{\sqrt{a}}$$

$$\frac{4}{\sqrt{a-\frac{x}{2}}} = kt + \frac{4}{\sqrt{a}}$$

$$\sqrt{a-\frac{x}{2}} = \frac{4}{kt+\frac{4}{\sqrt{a}}}$$

$$a - \frac{x}{2} = \left(\frac{4}{kt+\frac{4}{\sqrt{a}}}\right)^2$$

$$\frac{x}{2} = a - \left(\frac{4}{kt+\frac{4}{\sqrt{a}}}\right)^2$$

$$\boxed{x = 2a - 2\left(\frac{4}{kt+\frac{4}{\sqrt{a}}}\right)^2}$$