Math 2300: Calculus **Spring 2019**

Lecture 57: Wednesday April 10

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ebAssign due tonight.

57.1 Separable Equations

We've been talking a lot about differential equations. As a reminder, they look something like this:

$$\frac{dy}{dx} = 2y - x$$

So far, we can:

- Plot possible solution functions (slope fields)
- Estimate values of the solution function (Euler's method)

Today, we're going to talk about how to get exact solutions: exact families of solution functions, and exact functions (in cases where we have initial conditions).

57.1.1 An example of the process: without initial conditions

Solve the following differential equation:

$$\frac{dy}{dx} = 4x^3 y$$

Solution:
Split up the differential equation. Move the y-stuff to one side and the x-stuff to the other:

$$\frac{1}{y} dy = 4x^3 dx$$

Integrate both sides:

$$\int \frac{1}{y} dy = \int 4x^3 dx$$

$$\ln|y| = x^4 + C$$

Once you've integrated, solve for y:

$$|y| = e^{x^4+C}$$

Get rid of absolute value bars by putting $\pm$ on the other side:

$$y = \pm e^{x^4+C}$$

Using laws of exponents, you may more frequently see the answer written like this:

$$y = \pm e^{x^4+C} = \pm e^{x^4} e^C = \pm C' e^{x^4}$$

Since multiplying by e^C is equivalent to multiplying by some constant coefficient C'. Sloppily, you don't even need to switch C's:

$$y = \pm Ce^{x^4}$$

Note: This is a family of solutions: one solution for every possible C.

57.1.2 An example of the process: with initial conditions

Solve the initial value problem:

$$\frac{dy}{dx} = 4x^3 y$$
$$y(0) = 2$$

Solution:
The first step is always to solve the differential equation. Which we already did! Yay.

$$y = \pm Ce^{x^4}$$

Now, use the initial condition to solve for the C:

$$2 = \pm Ce^0 \Rightarrow \text{ Choose "+", so: } 2 = C$$

Plug it in and now we have ONE solution (no longer a family of solutions):

$$y = 2e^{x^4}$$

57.1.3 Example to try on your own

Solve the differential equation:

$$\frac{dx}{dt} - x = 1$$

HINT: Separate x and t:

$$\frac{dx}{dt} = x + 1$$
$$\frac{1}{x+1} dx = 1dt$$

Solution:
Now integrate:

$$\int \frac{1}{x+1} dx = \int 1dt$$
$$ln|x+1| = t + C$$
$$\text{Solve for } x:$$
$$|x+1| = e^{t+C}$$
$$\text{Get rid of absolute value bars:}$$
$$x + 1 = \pm Ce^t$$
$$x = \pm Ce^t - 1$$

Notice: Separating x and y will be the hardest part. The goal is to have dx connected with the x-stuff by multiplication, and the dy connected with the y-stuff by multiplication. It takes practice.

57.1.4 How to recognize separable equations

You **must** be able to separate the x's and y's using multiplication and division. Sometimes this can be difficult:

$$\frac{dy}{dx} = x + y$$

There's no way to immediately separate the x's and y's. Try it!
We need substitution to solve this differential equation.
Let $u = x + y$. Then

$$\frac{dy}{dx} = u$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx} = 1 + u$$

Working with:

$$\frac{du}{dx} = 1 + u$$

Separate the variables and integrate as usual:

$$\begin{aligned} \frac{1}{1+u} du &= 1dx \\ \int \frac{1}{1+u} du &= \int 1dx \\ \ln|1+u| &= x + C \\ \ln|x+y| &= x + C \\ |x+y| &= e^{x+C} \\ x + y &= \pm Ce^x \\ y &= \pm Ce^x - x \end{aligned}$$

57.1.5 Another Example

Solve the differential equation with initial value:

$$\frac{du}{dt} = u + ut,\ u(0) = 5$$

Solution:

$$\begin{aligned}
\frac{du}{dt} &= u + ut \\
\frac{du}{dt} &= u(1+t) \\
\frac{1}{u}du &= (1+t)dt \\
\int \frac{1}{u}du &= \int (1+t)dt \\
\ln|u| &= t + \frac{t^2}{2} + C \\
|u| &= e^{t+\frac{t^2}{2}+C} \\
u &= \pm e^{t+\frac{t^2}{2}+C} \\
&\text{OPTION 1:} \\
u &= \pm e^C e^{t+\frac{t^2}{2}} \\
u &= \pm C e^{t+\frac{t^2}{2}} \text{ Note the new `}C\text{'} \\
u &= \pm C e^{t+\frac{t^2}{2}} \\
&\text{Solve for } C\text{:} \\
5 &= \pm C e^{0+0} \\
&\text{Since 5 is positive, we choose "+":} \\
5 &= C \\
&\text{And plug it in:} \\
u &= 5e^{t+\frac{t^2}{2}} \\
&\text{OPTION 2:} \\
u &= \pm e^{t+\frac{t^2}{2}+C} \\
&\text{Solve for } C\text{:} \\
5 &= \pm e^{0+0+C} \\
&\text{Since 5 is positive, we choose "+":} \\
5 &= e^C \\
\ln(5) &= C \\
&\text{And plug it in:} \\
u &= e^{t+\frac{t^2}{2}+\ln(5)}
\end{aligned}$$

The two options give the same solution:

$$u = e^{t+\frac{t^2}{2}+\ln(5)} = e^{t+\frac{t^2}{2}} e^{\ln(5)} = 5e^{t+\frac{t^2}{2}}$$

57.1.6 ANOTHER Example

Solve the differential equation with initial value:

$$\frac{dy}{dx} = xe^y,\ y(0) = 0$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= xe^y \\
e^{-y}dy &= xdx \\
\int e^{-y}dy &= \int xdx \\
-e^{-y} &= \frac{x^2}{2} + C \\
e^{-y} &= -\frac{x^2}{2} - C \\
&\text{Solve for } C\text{:} \\
e^0 &= -0 - C \\
1 &= -C \\
-1 &= C \\
&\text{Plug it in and solve for } y\text{:} \\
e^{-y} &= -\frac{x^2}{2} + 1 \\
-y &= \ln(-\frac{x^2}{2} + 1) \\
y &= -\ln(-\frac{x^2}{2} + 1)
\end{aligned}$$