Math 2300: Calculus	**Spring 2019**

Lecture 50: Monday April 1

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WebAssign due tonight

50.1 Review

50.1.1 Representing a Function as a Power Series

Sometimes, it's useful to confirm that a given function has a power series representation. It might seem counter-intuitive that a power series (an infinite sum!) would be easier to work with than some given function, but in theory this is often true. The fact that a funciton has a power series representation means it has a lot of other nice properties too, so being able to confirm that you have a power series representation is very useful.
We will learn one way to confirm that we have a power series representation: Using the geometric series formula:

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$$

instead this time we will be interested in turning an expression that looks like the right-hand side into its power series representation on the left-hand side.

50.1.1.1 Example 1:

Find a power series representation for the function $\frac{2}{1+x}$. For what values of x will this power series representation hold?
Solution:
Write it so that it looks more like $\frac{1}{1-x}$:

$$\frac{2}{1+x} = 2\frac{1}{1-(-x)}$$

Now let's go the other way with our formula:

$$2\frac{1}{1-(-x)} = 2\sum_{n=0}^{\infty}(-x)^n$$

Note that this series is only convergent when $|-x| < 1$, so only for $x \in (-1, 1)$.

50.1.1.2 Example 4:

Find a power series representation for the function $\frac{1}{(1-x)^2}$. For what values of x will this power series representation hold?

Solution:
We'll need to use a derivative for this one. Note that:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

We are very familiar with the power series representation for $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Plugging this in:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$$

Notice that we can switch from starting at $n = 0$ to starting at $n = 1$, because if we plug in $n = 0$, the entire term is 0.
This is a geometric series, so it converges when $|x| < 1$, which is when $x \in (-1, 1)$.

50.1.1.3 Example 5:

Find a power series representation for the function $\frac{2x^2}{1+x^3}$. For what values of x will this power series representation hold?
Solution:
Look for the geometric series:

$$\frac{2x^2}{1+x^3} = 2x^2\frac{1}{1-(-x^3)} = 2x^2\sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{3n+2}$$

This only converges if $|x^3| < 1$, so only when $x \in (-1, 1)$.
More examples are posted on my website notes from the day we went over this in class.

50.1.2 Taylor Series and Approximation of Error

The Taylor Series expansion gives us a way to express even more kinds of functions as power series: now they don't just have to look like a geometric series, we have a formula for finding a power series expansion, as long as f has enough derivatives:

$$f(x) = T(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)(x-a)^n}{n!}$$

This is useful for the same reason finding power series representations of functions is useful. However, getting a Taylor Series expansion isn't going to be helpful in finding function values, since finding the value a series converges to is quite difficult.
However, knowing the Taylor Series is there in the background can help us figure out how accurate a Taylor Polynomial is. (Recall: an nth Taylor polynomial just truncates the Taylor series at the nth degree term.)
Here's the theorem, called **Taylor's Inequality**:
Let $f(x)$ be a function, and let $T_n(x)$ be the nth degree Taylor Polynomial for $f(x)$ centered at $x = a$. If $f^{(n+1)}(x)$ is continuous and satisfies $|f^{(n+1)}(x)| \leq M$ for all values of x such that $|x - a| < d$ (i.e., x is less

than d away from where we center the Taylor Series), then the remainder $f(x) - T_n(x) = R_n(x)$ satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x such that $|x-a| < d$.

This property $|f^{(n+1)}(x)| \leq M$ is important. Not all of the functions we know have this property. For example, if $f(x) = x^2$, then $f'(x) = 2x$ does not have this property everywhere. But if we limit ourselves to an interval, say we do a Taylor series centered at $x = 0$ and then only use it in $[-1, 1]$, then $f'(x)$ is bounded on $[-1, 1]$: it's always ≤ 2.

50.1.2.1 Example

Consider $f(x) = e^x$:

(a) Find the Taylor Series for $f(x)$ centered at $x = 0$.

(b) Find the 4th degree Taylor Polynomial for $f(x)$ using part (a).

(c) Use $T_4(x)$ from part (b) to approximate $e^{0.1}$.

(d) How accurate is your approximation in part (c) guaranteed to be?

Solution:

(a) Let's use the table method again, and see if we can find the pattern.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}(x-0)^n$
0	e^x	1	$\frac{1}{0!} = 1$	1
1	e^x	1	$\frac{1}{1!} = 1$	(x-0)
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$	$\frac{1}{2}(x-0)^2$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$	$\frac{1}{6}(x-0)^3$
4	e^x	1	$\frac{1}{4!} = \frac{1}{24}$	$\frac{1}{24}(x-0)^4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	e^x	1	$\frac{1}{n!}$	$\frac{1}{n!}(x-0)^n$

We've done this previously, so we can also just look it up:

$$e^x = T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

NOTE: When we plug in values of x into e^x, we can do the same thing on the other side. For example:

$$e^3 = T(3) = \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

Some questions may ask you to "evaluate" the series $\sum_{n=0}^{\infty} \frac{3^n}{n!}$. What they mean is really just "recognize that $\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3$". I will include a table of useful-to-recognize Maclaurin series at the end of these notes.

(b) Just using the terms up to degree 4, we get:

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

(c) Now just plug in $x = .1$:

$$T_4(0.1) = 1 + .1 + \frac{.1^2}{2} + \frac{.1^3}{6} + \frac{.1^4}{24} \approx 1.10517083...$$

(d) Using Taylor's Inequality, we know:

$$|R_4(x)| \leq \frac{M}{5!}(x-0)^5$$

for $|x-0| < d$, where $|f^{(5)}(x)| \leq M$ for $|x-0| < d$. Note that $f^{(5)}(x) = e^x$. We are estimating at $x = 0.1$, so we are looking at $|x| \leq 0.1$. How can we simply bound e^x in an interval around 0 that includes 0.1? In this range, we can be sure that e^x is less than 2. For $|x| \leq 0.1$, I can be confident that $e^x < 2$. Why? Because $e^1 = 2.7...$, and $e^0 = 1$, so I believe $e^.1 < 2$. So use this value for M:

$$|R_4(x)| \leq \frac{2}{5!}x^5$$

$$\Rightarrow |R_4(.1)| \leq \frac{2}{5!}.1^5 \approx 0.000000167...$$

Since we have calculators, we can check that this is true:

$$e^.1 - 1.10517083... \approx 0.000000088...,$$

so we can see that we are within 0.000000167 of the true answer with our estimate using a 4th degree polynomial.

50.1.3 Maclaurin Series to Recognize

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Useful Maclaurin Series

50.1.4 Taylor Series Example

Find the Maclaurin Series expansion of $f(x) = xe^{2x}$. Determine the radius of convergence. Start by making a table: Once you see the pattern, you have it:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}x^n$
0	xe^{2x}	0	0	0
1	$e^{2x}(2x+1)$	1	1	x
2	$2e^{2x}(2x+2)$	4	$\frac{4}{2!} = 2$	$2x^2$
3	$4e^{2x}(2x+3)$	12	$\frac{12}{3!} = 2$	$2x^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$2^{k-1}e^{2x}(2x+k)$	$2^{k-1}k$	$\frac{2^{k-1}k}{k!} = \frac{2^{k-1}}{(k-1)!}$	$\frac{2^{k-1}k}{k!}x^k = \frac{2^{k-1}}{(k-1)!}x^k$

$$f(x) = T(x) = \sum_{n=1}^{\infty} \frac{2^{n-1}x^n}{(n-1)!}$$

To determine the radius of convergence, use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{2^n x^{n+1}}{n!}\cdot\frac{(n-1)!}{2^{n-1}x^n}\right| &= \lim_{n\to\infty}\left|\frac{2x}{n}\right| \\ &= 2\lim_{n\to\infty}\frac{|x|}{n} \\ &= 0 < 1 \text{ for all } x\end{aligned}$$

so the interval of convergence is all real numbers, and the radius of convergence is ∞.