Math 2300: Calculus **Spring 2019**

Lecture 49: Friday March 22

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WebAssign due tonight

49.1 Warm Up

Find the Taylor Series expansion for $f(x) = \sin(2x)$ about $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}(x-0)^n$
0	$\sin(2x)$	0	0	0
1	$2\cos(2x)$	2	2	$2x$
2	$-4\sin(2x)$	0	0	0
3	$-8\cos(2x)$	-8	$\frac{-8}{3!}$	$\frac{-4}{3}x^3$
4	$16\sin(2x)$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

From the table, we can recognize a pattern:

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

If you remembered the Taylor Series we had for $\sin(x)$, note that this is the series we get when we plug $2x$ into that series.

49.2 8.7: Taylor Series, Part II

We've defined Taylor Series, but we've yet to explore when these series are valid representations for the corresponding function. Today we will start to talk about that, and also look at the error involved.

49.2.1 Convergence to Taylor Series

This is in general quite difficult. I just want you to be aware of it, but we will usually just assume we have what we need in order to guarantee "$f(x) = T(x)$". Let $T_n(x)$ denote the nth degree Taylor Polynomial (so, finite-degree), and let $T(x)$ denote the Taylor Series of $f(x)$. Then let $R_n(x)$ denote the error of the nth degree Taylor polynomial approximation: $R_n(x) = f(x) - T_n(x)$.
Suppose that $f(x) = T_n(x) + R_n(x)$. Then, if

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < C$, then $f(x) = T(x)$.
The issue is not this statement, but showing that $\lim_{n\to\infty} R_n(x) = 0$ is difficult. We will usually just assume it, without saying anything.

49.2.2 Approximation of Error

Let's return to our original motivation for this stuff: Function approximation. The Taylor Series isn't going to be helpful in finding function values, since finding the value a series converges to is quite difficult.
However, knowing the Taylor Series is there in the background can help us figure out how accurate a Taylor Polynomial is.
Here's the theorem, called **Taylor's Inequality**:
Let $f(x)$ be a function, and let $T_n(x)$ be the nth degree Taylor Polynomial for $f(x)$ centered at $x = a$. If $f^{(n+1)}(x)$ is continuous and satisfies $|f^{(n+1)}(x)| \leq M$ for all values of x such that $|x - a| < d$ (i.e., x is less than d away from where we center the Taylor Series), then the remainder $f(x) - T_n(x) = R_n(x)$ satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x such that $|x - a| < d$.

49.2.2.1 Example

Consider $f(x) = e^x$:

(a) Find the Taylor Series for $f(x)$ centered at $x = 0$.

(b) Find the 4th degree Taylor Polynomial for $f(x)$ using part (a).

(c) Use $T_4(x)$ from part (b) to approximate $e^{0.1}$.

(d) How accurate is your approximation in part (c) guaranteed to be?

Solution:

(a) Let's use the table method again, and see if we can find the pattern. We've done this previously, so we can also just look it up:

$$e^x = T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(b) Just using the terms up to degree 4, we get:

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

(c) Now just plug in $x = .1$:

$$T_4(0.1) = 1 + .1 + \frac{.1^2}{2} + \frac{.1^3}{6} + \frac{.1^4}{24} \approx 1.10517083...$$

(d) Using Taylor's Inequality, we know:

$$|R_4(x)| \leq \frac{M}{5!}(x-0)^5$$

for $|x - 0| < d$, where $|f^{(5)}(x)| \leq M$ for $|x - 0| < d$. Note that $f^{(5)}(x) = e^x$. We are estimating at $x = 0.1$, so we are looking at $|x| \leq 0.1$. How can we simply bound e^x in an interval around 0 that includes 0.1?

In this range, we can be sure that e^x is less than 2. For $|x|\leq 0.1$, I can be confident that $e^x < 2$. Why? Because $e^1 = 2.7...$, and $e^0 = 1$, so I believe $e^{\cdot}1 < 2$. So use this value for M:

$$|R_4(x)|\leq \frac{2}{5!}x^5$$

$$\Rightarrow |R_4(.1)|\leq \frac{2}{5!}.1^5 \approx 0.000000167...$$

Since we have calculators, we can check that this is true:

$$e^{\cdot}1 - 1.10517083... \approx 0.000000088...,$$

so we can see that we are within 0.000000167 of the true answer with our estimate using a 4th degree polynomial.