Math 2300: Calculus **Spring 2019**

Lecture 47: Wednesday March 20

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WebAssign due Tonight

47.1 Warm-up

Have the students find a 3rd degree Taylor polynomial of a function of your choice.

47.2 8.7: Taylor Series, Part I

We know how to get a power series representation for functions that "look like" a geometric series sum. But what about other functions?

Recall what we did the week after Exam 2: Taylor Polynomials! These polynomials were a great way to approximate function values. The higher the degree, the better the approximation. If we turn them into power series ("infinite degree" polynomials), they are perfect!

47.2.1 Definition:

The **Taylor Series** for $f(x)$ centered at $x = a$ is defined:

$$T(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(k)}(a)(x-a)^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

When we center it at $x = 0$, it is called a **MacLaurin series** (sometimes... you can also just say "Taylor Series centered at $x = 0$"). Taylor polynomials approximate the function ($T_n(x) \approx f(x)$), but the Taylor series *is* the function: $T(x) = f(x)$, when it exists.

Notice that we've made a few assumptions in even writing down $T(x)$:

- $f(x)$ needs to be infinitely-many-times differentiable at the point $x = a$ where we center the Taylor series. Nice functions will have this property, but not all functions have this property.

- We've also assume that $f(x)$ does indeed have some power series representation. This one is more subtle. When $f(x)$ has a power series representation, it is given by $T(x)$.

47.2.2 Nice Examples

In some cases, it's possible to get a nice general form for the coefficients of $T(x)$. Let's start out by looking at a few of these examples:

47.2.2.1 Example 1:

Find the Taylor Series expansion for $f(x) = e^x$ about $x = 0$.
Solution:
The chart method we used in the past will still be helpful. Look for patterns and try to get a general form for the sum. Notice the 'pattern' of $f^{(n)}(0)$ values. They are all the same! This makes the Taylor Series

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}(x-0)^n$
0	e^x	1	$\frac{1}{0!} = 1$	1
1	e^x	1	$\frac{1}{1!} = 1$	(x-0)
2	e^x	1	$\frac{1}{2!} = \frac{1}{2}$	$\frac{1}{2}(x-0)^2$
3	e^x	1	$\frac{1}{3!} = \frac{1}{6}$	$\frac{1}{6}(x-0)^3$
4	e^x	1	$\frac{1}{4!} = \frac{1}{24}$	$\frac{1}{24}(x-0)^4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	e^x	1	$\frac{1}{n!}$	$\frac{1}{n!}(x-0)^n$

very simple to write down:

$$T(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We won't always luck out ($e^x + C$ is actually the only function that is its own derivative), but there are still nice patterns out there.

47.2.2.2 Example 2:

Find the Taylor Series expansion for $f(x) = \cos(x)$ about $x = 0$.
Solution:
Again, start with the chart method: Notice the pattern this time: we only have even powers of x, and they

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(x)}{n!}$	$\frac{f^{(n)}(x)}{n!}(x-0)^n$
0	$\cos(x)$	1	1	1
1	$-\sin(x)$	0	0	0
2	$-\cos(x)$	-1	$\frac{-1}{2}$	$\frac{-1}{2}(x-0)^2$
3	$\sin(x)$	0	0	0
4	$\cos(x)$	1	$\frac{1}{4!}$	$\frac{1}{4!}(x-0)^4$
5	$-\sin(x)$	0	0	0

alternate positive and negative. Think about how to represent something that alternates like that in general. If n is counting integers, we need to have $2n$ in the exponents. How to fix the powers of (-1)? Don't use $2n$!

$$T(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

47.2.3 Less Nice Examples

So far, we've been able to get a general form for the coefficients of the Taylor series expansion pretty easily. This will not always be the case...

47.2.3.1 Example 1:

Find the first four nonzero terms of the Taylor Series expansion for $f(x) = e^x \cos(x)$ about $x = 0$.
Solution 1:
We can again use a chart, but this time there is no hope for a nice general form. So we can get the first few

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(x)}{n!}$	$\frac{f^{(n)}(x)}{n!}(x-0)^n$
0	$e^x \cos(x)$	1	1	1
1	$-e^x \sin(x) + e^x \cos(x)$	1	1	$(x-0)^1$
2	$-2e^x \sin(x)$	0	0	0
3	$-2e^x \sin(x) - 2e^x \cos(x)$	-2	$\frac{-2}{6}$	$\frac{-1}{3}(x-0)^3$
4	$-4e^x \cos(x)$	-4	$\frac{-4}{24}$	$\frac{-1}{6}(x-0)^4$

terms:

$$T(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

textbfSolution 2:
Instead of doing a whole new chart, we can actually mutiply together the Taylor series we got from the first two examples we did! We just need to make sure we take the right numbers of terms:

$$e^x \cos(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)$$

Think of this multiplication as distributing the second series to the terms of the first:

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + x\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \frac{x^2}{2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \frac{x^3}{6}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \cdots$$

Now collect the terms of like degrees:

For 4 nonzero terms, we know we need up to degree-4, so make sure to get extra terms if anything:

$$= 1 + x + x^2\left(\frac{-1}{2} + \frac{1}{2}\right) + x^3\left(\frac{-1}{2} + \frac{1}{6}\right) + x^4\left(\frac{1}{24} - \frac{1}{4}\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

So that gives us two ways to do it!

47.2.4 Another Important Nice Example

We found the Taylor Series expansion for $\cos(x)$, so we would be remiss to not include $\sin(x)$ too: This time

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(0)}{n!}(x-0)^n$
0	$\sin(x)$	0	0	0
1	$\cos(x)$	1	1	x
2	$-\sin(x)$	0	0	0
3	$-\cos(x)$	-1	$\frac{-1}{3!}$	$\frac{-1}{6}x^3$
4	$\sin(x)$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

we need a way to access only odd degree terms. To do this, use $2n+1$, much like how we used $2n$ when we needed to access even powers. Then make the power of -1 work out as needed: in this case (as in the last), we still need the terms to alternate positive, negative, positive,...:

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$