**Math 2300: Calculus** **Spring 2019**

# Lecture 46: Tuesday March 19

*Lecturer: Sarah Arpin*

## 46.1 8.6: Representing Functions with Power Series

Today we will learn to use the fact that we know the value of a convergent geometric series to write power series as functions. We had an introduction to this yesterday, so 46.1.1 and 46.1.2 should go quickly. Same with Examples 1 and 2. Spend more time on Examples 3 and 4.

### 46.1.1 Recall Geometric Series Information:

If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges. Recall that we have the formula:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

What does this tell us about $\sum_{n=0}^{\infty} ar^n$?

By writing out the first few terms of the series $\sum_{n=0}^{\infty} ar^n$ and the first few terms of the series $\sum_{n=1}^{\infty} ar^{n-1}$ to justify

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Note that it doesn't matter where you index your sum: If it's a geometric series, then $a$ is always the first term and $r$ is always the part that's being raised to the $n$ power.

### 46.1.2 Recall Power Series Information:

The power series $\sum_{n=0}^{\infty} x^n$ was one of the first ones we looked at. Remember how we looked at it and recognized that it is a geometric series with $r = x$, so it converges precisely when $x \in (-1, 1)$.

When we assume $x \in (-1, 1)$, how can we use the formula $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$?

*Have the students think about this for a little bit. Remind them that a is always just the first term of the series and r is always the part being raised to the n-power.*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

So we have this formula for the power series $\sum_{n=0}^{\infty} x^n$, provided $x \in (-1, 1)$.

We call $\sum_{n=0}^{\infty} x^n$ the **power series representation** of the function $\frac{1}{1-x}$.

**Subtlety:** Note that we can plug in any $x \neq 1$ into the formula $\frac{1}{1-x}$, but it's only equal to the power series at that $x$-value for $x$'s in $(-1,1)$. This is a big deal for mathematicians! The formula on the right has a larger domain than the power series function on the left, but they are only equal for $x$'s in $(-1,1)$. This is a first peek at a strange analysis topic called "analytic continuation". For Calc 2, you don't need to know anything about this besides the fact that the formula only holds when the power series converges.

## 46.1.3 Back to the point

Functions that "look like" $\frac{1}{1-x}$ can be written as power series!
Let's practice going from function to power series. The first two should be familiar:

#### 46.1.3.1 Example 1:

Find a power series representation for the function $\frac{2}{1+x}$. For what values of $x$ will this power series representation hold?
**Solution:**
Write it so that it looks more like $\frac{1}{1-x}$:

$$\frac{2}{1+x} = 2\frac{1}{1-(-x)}$$

Now let's go the other way with our formula:

$$2\frac{1}{1-(-x)} = 2\sum_{n=0}^{\infty}(-x)^n$$

Note that this series is only convergent when $|-x| < 1$, so only for $x \in (-1,1)$.

#### 46.1.3.2 Example 2:

Find a power series representation for the function $\frac{2}{3+2x}$. For what values of $x$ will this power series representation hold?
**Solution:**
Look for the geometric series:

$$\frac{2}{3+2x} = 2 \cdot \frac{1}{3(1+\frac{2}{3}x)} = \frac{2}{3} \cdot \frac{1}{1-(\frac{-2}{3}x)} = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{-2}{3}x\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}x\right)^{n+1}$$

This only converges if $|\frac{2}{3}x| < 1$, which is when $x \in (-3/2, 3/2)$.
*Review solving inequalities with absolute values*: Note that $|\frac{2}{3}x| < 1$ turns into $\frac{2}{3}x < 1$ AND $\frac{2}{3}x > -1$, so solving this yields $x < 3/2$ AND $x > -3/2$.

#### 46.1.3.3 Example 3:

Find a power series representation for the function $\ln(5-x)$. What is the interval of convergence?
**Solution:**
We'll need to use an integral for this one. Note that:

$$-\int \frac{1}{5-x}dx = \ln(5-x) + C.$$

*Note: I'm putting the constant of integration on the right so that we can worry about it later.* It's easy to find a power series representation for $\frac{1}{5-x}$:

$$\frac{1}{5-x} = \frac{1}{5}\cdot\frac{1}{1-\frac{x}{5}} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$$

Plugging this in, we can integrate to finish the job:

$$\begin{aligned}
\ln(5-x) + C &= -\int \frac{1}{5-x}dx \\
&= -\int \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n dx \\
&= \frac{-1}{5}\int \sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n dx \\
&= \frac{-1}{5}\sum_{n=0}^{\infty}\int \frac{x^n}{5^n}dx \\
&= \frac{-1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^n} \\
&= \sum_{n=0}^{\infty}\frac{-x^{n+1}}{(n+1)5^{n+1}}
\end{aligned}$$

In summary, we have:

$$\ln(5-x) + C = \sum_{n=0}^{\infty}\frac{-x^{n+1}}{(n+1)5^{n+1}}$$

Now that we're nearly there, we just need to worry about the constant of integration. To figure out what it must be, plug in a value for $x$ on both sides. $x=0$ will be easy:

$$\ln(5-0) + C = \sum_{n=0}^{\infty}\frac{-0^{n+1}}{(n+1)5^{n+1}}$$

$$\ln(5) + C = 0$$

$$C = -\ln(5)$$

So, in summary, we have:

$$\ln(5-x) - \ln(5) = \sum_{n=0}^{\infty}\frac{-x^{n+1}}{(n+1)5^{n+1}}$$

Solve for $\ln(5-x)$ to give our final answer:

$$\ln(5-x) = \ln(5) + \sum_{n=0}^{\infty}\frac{-x^{n+1}}{(n+1)5^{n+1}}$$

To find the interval of convergence, note that the power series we end up with is a geometric series with $r = \frac{x}{5}$, so it converges for $x \in (-5, 5)$.

#### 46.1.3.4 Example 4:

Find a power series representation for the function $\frac{1}{(1-x)^2}$. For what values of $x$ will this power series representation hold?
**Solution:**
We'll need to use a derivative for this one. Note that:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

We are very familiar with the power series representation for $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Plugging this in:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$

Notice that we can switch from starting at $n = 0$ to starting at $n = 1$, because if we plug in $n = 0$, the entire term is 0.
This is a geometric series, so it converges when $|x| < 1$, which is when $x \in (-1, 1)$.

#### 46.1.3.5 Example 5:

Find a power series representation for the function $\frac{2x^2}{1+x^3}$. For what values of $x$ will this power series representation hold?
**Solution:**
Look for the geometric series:

$$\frac{2x^2}{1+x^3} = 2x^2 \frac{1}{1-(-x^3)} = 2x^2 \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{3n+2}$$

This only converges if $|x^3| < 1$, so only when $x \in (-1, 1)$.