Math 2300: Calculus **Spring 2019**

Lecture 44: Friday March 14

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44.1 8.5: Power Series

We've been talking about polynomials (Taylor polynomials specifically). Before the exam, we talked about series. Now, we'll talk about those two concepts together.

A **power series** is an "infinite degree polynomial". In other words, it's a series where we have a variable x (a variable besides the iteration variable). In general, it looks like:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

A particular example:

$$\sum_{n=0}^{\infty} n x^n = 0 + x + 2x^2 + 3x^3 + \cdots$$

You should think of a power series as a function, where the input is some value that we provide for x, and the output is a series (a sum of numbers).

Small note: Notice that you can start indexing your sum at any non-negative integer. You can start at 5, you'll just miss out on certain powers of x:

$$\sum_{n=5}^{\infty} n x^n = 5x^5 + 6x^6 + 7x^7 + \cdots$$

But we **cannot** index at negative numbers, because then we don't get a polynomial anymore:

$$\sum_{n=-1}^{\infty} n x^n = \frac{-1}{x} + 0 + x + 2x^2 + 3x^3 + \cdots$$

We talked a lot about determining when a series converges. For example:

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

But, if we have a power series with a variable x, we can ask **"for what values of x does the series converge?"** The two series above can be thought of just different evaluations of the function:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x.$$

(Note that this is not a power series (it's not an "infinite-degree" polynomial), but it illustrates the point we are considering at the moment, so we will consider it just as motivation). We know that when $x \leq 1$, the series diverges, and when $x > 1$ the series converges. Let's generalize this to less obvious scenarios. To find the answer, we will use the **ratio test.**

44.1.1 Introductory Example:

For what values of x does the power series $\sum_{n=0}^{\infty} x^n$ converge?
Solution:
This one is a softball. We can recognize this as a geometric series with $r = x$, so it converges when $x \in (-1, 1)$ by the geometric series test.
Even though we know the answer, let's practice using the ratio test to answer these types of questions:
By the ratio test, we need $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ to guarantee convergence:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \\ &= \lim_{n\to\infty} |x| \\ &= |x|\end{aligned}$$

So we require $|x| < 1$, which means $x \in (-1, 1)$. This guarantees absolute convergence, even.

Remember that the ratio test is uncertain when the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. So we test these boundary conditions individually:
Boundary Case 1: $x = 1$:

$$\sum_{n=0}^{\infty} 1^n, \text{ diverges by divergence test.}$$

Boundary Case 2: $x = -1$

$$\sum_{n=0}^{\infty} (-1)^n, \text{ diverges by divergence test.}$$

So the **interval of convergence** of this power series is $(-1, 1)$. We say that the **radius of convergence** is 1, because 1 is the radius of the interval $(-1, 1)$.

44.1.2 Example 1:

What is the radius of convergence of the power series $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$?
Solution:
First, figure out how to write this power series using sigma notation. We should recognize the denominators as factorials:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now, investigate using the ratio test. We need $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, and then we will investigate whatever endpoints we have individually:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| \\ &= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| \\ &= 0\end{aligned}$$

Since $0 < 1$ for all x, this power series has an infinite radius of convergence: $(-\infty, \infty)$. No need to check the boundary numbers when there's no boundary!

44.1.3 Example 2:

Find the interval and radius of convergence of the series:

$$\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n$$

Solution:
This one is not geometric, so we MUST use the ratio test, and then investigate the boundary separately. By the ratio test, we need:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{2^{n+1}(4x-8)^{n+1}}{(n+1)}\cdot\frac{n}{2^n(4x-8)^n}\right| &= \lim_{n\to\infty}\left|\frac{2(4x-8)}{(n+1)}\cdot\frac{n}{1}\right| \\ &= 2|(4x-8)|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| \\ &= |8x-16|\end{aligned}$$

So we need $|8x-16| < 1$ (for the Ratio Test to give us the conclusion "absolutely convergent), and then we need to check the boundary conditions. (For a review of absolute value inequalities, see HERE).

$$-1 < 8x - 16 < 1$$

$$15 < 8x < 17$$
$$\frac{15}{8} < x < \frac{17}{8}$$

We can plug in $x = 17/8$ and $x = 15/8$ individually to check these boundary conditions now.
For $x = 17/8$:

$$\sum_{n=0}^{\infty}\frac{2^n}{n}(4(17/8)-8)^n = \sum_{n=0}^{\infty}\frac{2^n}{n}((17/2)-8)^n = \sum_{n=0}^{\infty}\frac{2^n}{n}(1/2)^n = \sum_{n=0}^{\infty}\frac{1}{n},$$

which diverges by p-series test with $p = 1 \leq 1$.
For $x = 15/8$:

$$\sum_{n=0}^{\infty}\frac{2^n}{n}(4(15/8)-8)^n = \sum_{n=0}^{\infty}\frac{2^n}{n}((15/2)-8)^n = \sum_{n=0}^{\infty}\frac{2^n}{n}(-1/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n},$$

which is the alternating harmonic series, and we know this one converges (quick alternating series test proves it). So our interval of convergence is: $[15/8, 17/8)$. The radius of convergence is half the length of this interval, so it is:

$$\text{radius} = \frac{1}{2}\left(\frac{17}{8} - \frac{15}{8}\right) = \frac{1}{8}$$

44.1.4 Example 3:

Find the interval and radius of convergence of the series:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{2^n}$$

Solution:
Two options for how to do this one: We can recognize it as a geometric series and force $|r| < 1$, or we can do

our ratio test and boundary check procedure as usual.
Since we know it's a geometric series, let's do that procedure:

$$\sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{(-1)(x-3)}{2}\right)^n$$

So $r = \left(\frac{(-1)(x-3)}{2}\right)$. Solving the absolute value inequality:

$$|r| < 1$$
$$\left|\left(\frac{(-1)(x-3)}{2}\right)\right| < 1$$
$$|x-3| < 2$$

This corresponds to the double inequality:

$$-2 < x - 3 < 2$$

Solving this inequality, we get $1 < x < 5$. so the interval of convergence is $(1, 5)$.
The radius of convergence is half the length of this interval, so it is 2.
It was nice that we recognized this as a geometric series, because it means we don't have to test the endpoints.
There's no "inconclusive" part in the geometric series test: it converges for $|r| < 1$ and diverges for $|r| \geq 1$.