Math 2300: Calculus **Spring 2019**

Lecture 42: Wednesday March 13

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42.1 8.7: Taylor Polynomials (Continued)

Last class, we introduced Taylor polynomials as a way to approximate complicated functions. Let's formalize that now:

42.1.1 Definition

$T_n(x)$, the nth degree Taylor polynomial for $f(x)$ centered at $x = a$ is defined:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We might want such a polynomial because polynomials are easier to compute with than other classes of functions. For example, polynomials are nicer than exponentials, or logs, etc.
If $a = 0$, we call $T_n(x)$ a Maclaurin polynomial

42.1.2 Noteworthy Property

- Notice that $T_n(a) = f(a), T_n'(a) = f'(a), ..., T_n^{(n)}(a) = f^{(n)}(a)$.

42.1.3 Example 1

(a) Find $T_6(x)$ the 6th degree Taylor polynomial for $f(x) = \cos(x)$ centered at $x = 0$.

(b) Use your polynomial to estimate $\cos(5°)$.

(c) $\cos(x)$ is an even function (i.e., $\cos(-x) = \cos(x)$). Is $T_6(x)$?

Solution:

(a) Let's make a table to organize the derivative information: Using these to create our polynomial:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(x)$	$\cos(0) = 1$
1	$-\sin(x)$	$-\sin(0) = 0$
2	$-\cos(x)$	$-\cos(0) = -1$
3	$\sin(x)$	$\sin(0)=0
4	$\cos(x)$	$\cos(0) = 1$
5	$-\sin(x)$	$-\sin(0) = 0$
6	$-\cos(x)$	$-\cos(0) = -1$

$$T_6(x) = 1 + 0 \cdot x + \frac{-1 \cdot x^2}{2} + \frac{0 \cdot x^3}{3!} + \frac{1 \cdot x^4}{4!} + \frac{0 \cdot x^5}{5!} + \frac{-1 \cdot x^6}{6!}$$

Simplifying:

$$T_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$

(b) To use it to approximate $\cos(5)$, we just need to plug in 5. But since we are using a trig function, **we must convert to radians**: $5 = \pi/36$:

$$T_6(5) = 1 - \frac{1}{2}(\pi/36)^2 + \frac{1}{24}(\pi/36)^4 - \frac{1}{720}(\pi/36)^6 \approx 0.99619$$

This is a pretty good estimate! $\pi/36$ isn't far from zero, so this makes sense.

(c) Yes! To see that, note that $T_6(x)$ only has even powers of x, so $T_6(-x) = T_6(x)$.

42.1.4 Example 2

(a) What is $T_n(x)$, the nth degree Taylor polynomial for $f(x) = \ln(x)$ centered at $x = 1$?

(b) Estimate $\ln(2)$ using $T_4(x)$.

(c) What could you do to improve your estimate in part (b)?

Solution:

(a) We can make another table to get organized, and let's start looking for the pattern in the derivatives: If

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	$\ln(1) = 0$
1	$\frac{1}{x}$	1
2	$\frac{-1}{x^2}$	-1
3	$\frac{2}{x^3}$	2
4	$\frac{-6}{x^4}$	-6
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(-1)^{n-1}(n-1)!}{x^n}$	$(-1)^{n-1}(n-1)!$

you didn't recognize the pattern at $T_4(x)$, just keep going. Look for common patterns: Notice that we keep multiplying by the exponent, so that generates the $(n-1)!$. Notice that the even-power terms are negative, and the odd-power terms are positive, so that gives us the $(-1)^{n-1}$ part. **It takes practice to be able to recognize these patterns.**

Using this chart to write down the polynomial:

$$T_n(x) = 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

$$T_n(x) = (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{-1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

(b) Cut this off at $n = 4$ and plug in $x = 2$:

$$T_4(2) = (2-1) + \frac{-1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 + \frac{-1}{4}(2-1)^4 \approx 0.583333$$

(c) It isn't a great estimate: $\ln(2) \approx 0.693147$. How can we do better?

- You could use a higher degree Taylor polynomial (T_6(x), for example).
- You could do a Taylor polynomial centered at $x = e$ instead, since e is closer to 2 than 1 is.

42.1.5 A Look Ahead

If we construct an "infinite degree" Taylor polynomial, a lot of the time we can get *precisely* the function that we want. Now, polynomials are only allowed to be finite degree. So this is a stretch. But using something called a **power series**, which we will introduce on Friday, this is possible.

How would we write out the "infinite degree" Taylor polynomial from the example above? Using sigma notation!

$$T_n(x) = (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{-1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$

$$T_\infty(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$$

If we plug in $x = 2$ now, we see we get a convergent series (by the alternating series test). It's still hard to show what it converges to...but trust me, it converges to $\ln(2)$.