Math 2300: Calculus **Spring 2019**

Lecture 37: Wednesday March 6

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37.1 Alternating Series Test: Remainder Estimate

37.1.1 Introduction

When we talked about the alternating series test, a lot of you asked "But how do we know what it converges to?"

- We estimate these sums by looking at where the partial sums go towards.
- It's difficult to find an exact answer for what these.
- But it's not difficult to give a bound on how far off a partial-sum-guess might be!

Let's find a theorem about how far off a particular partial sum estimate could be from the actual sum. We will do this for alternating series, which have a particularly nice result about errors of estimates.

37.1.2 Investigation

Let's take a familiar alternating series that we know converges. Use the 3rd, 4th, 5th, and 6th partial sums to estimate the value of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Solution:

$$\text{3rd partial sum: } \frac{-1}{1}+\frac{1}{2}+\frac{-1}{3}=\frac{-5}{6}\approx -0.8333$$
$$\text{4th partial sum: } \frac{-1}{1}+\frac{1}{2}+\frac{-1}{3}+\frac{1}{4}=\frac{-7}{12}\approx -0.5833$$
$$\text{5th partial sum: } \frac{-1}{1}+\frac{1}{2}+\frac{-1}{3}+\frac{1}{4}+\frac{-1}{5}=\frac{-47}{60}\approx -0.7833$$
$$\text{6th partial sum: } \frac{-1}{1}+\frac{1}{2}+\frac{-1}{3}+\frac{1}{4}+\frac{-1}{5}+\frac{1}{6}=\frac{-37}{60}\approx -0.6167$$

Graph these on a number line to see what's happening.
We are closing in on a solution! But how far away are we on any given step?
Let S denote the actual sum, let S_N denote the Nth partial sum.

$$|S - S_N| \leq |S_{N+1} - S_N| = |a_{N+1}|$$

The error for the estimate of an alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ at the Nth partial sum is $\leq |a_{N+1}|$.

How can we use this?

37.1.3 Example

What is the maximum possible error for the 10th partial sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{\sqrt[3]{n^2+4}}$$

Solution:
By the alternating series remainder theorem, the error of the 10th partial sum is bounded by the 11th term:

$$|S - S_{10}| \leq |a_{11}| = \left| \frac{(-1)^{11} \cdot 11}{\sqrt[3]{121+4}} \right| = \frac{11}{5}$$

37.1.4 Example

How many terms do you have to add up to estimate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

to within 0.0001 accuracy?
Solution:
Let's see. If we do the Nth partial sum, we will get an estimate that is less than or equal to $|a_{N+1}|$ off from the true value. This means we need $|a_{N+1}| \leq 0.0001$. Write this inequality down and solve for N:

$$\begin{aligned} |a_{N+1}| &\leq 0.0001 \\ \frac{1}{N^2} &\leq \frac{1}{10000} \\ 10000 &\leq N^2 \\ 100 &\leq N \end{aligned}$$

So if we do the $N = 100$th partial sum, we can be sure our estimate will be within 0.0001 of the true value.

37.2 Integral Test Remainder Estimate

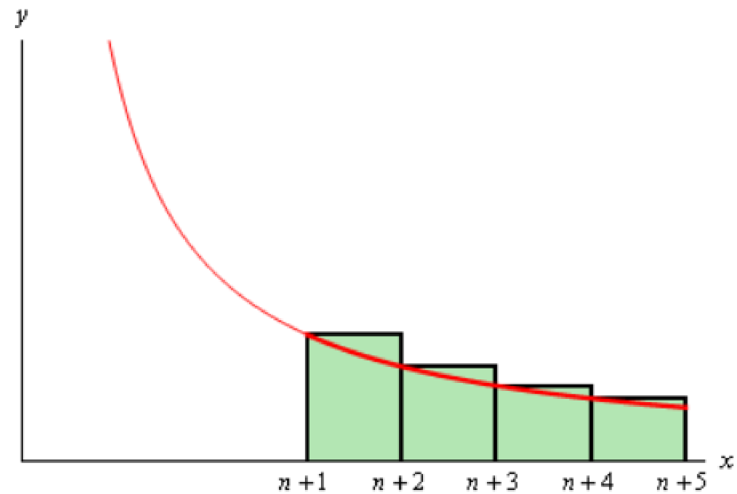
Recall the hypotheses required for the convergence of a series $\sum_{n=1}^{\infty} a_n$ to be determined by the integral test:

- We need to be able to define a function f such that $f(n) = a_n$ that is **continuous** on $[1, \infty)$,
- $f(x)$ needs to be **positive** on $[1, \infty)$,
- $f(x)$ needs to be **decreasing** on $[1, \infty)$.

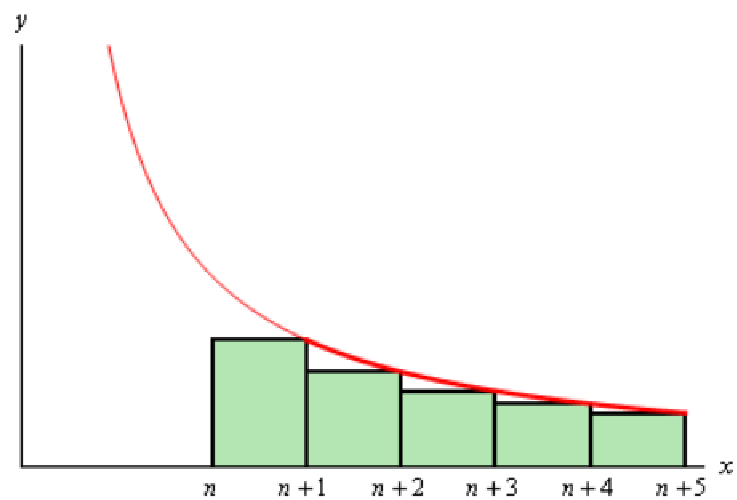
When these hypotheses are satisfied, we can conclude that $\sum_{n=1}^{\infty} a_n$ converges **if and only if** $\int_1^{\infty} f(x)dx$ converges.

When we first introduced the Integral Test, we gave a visual reason why it would work - Riemann sums! In the setting of the integral test, the function that gives our series at positive integer points is continuous positive and decreasing, so if we draw the Riemann sums, they have to look a certain way (sketch on board).

Let's say we use N rectangles to estimate our error (this would be an Nth partial sum). If we keep going with the left-endpoint rectangles, we can see that the partial sum is off by at least $\int_{n+1}^{\infty} f(x)dx$: The rectangles overlap the curve:



If we draw instead right-endpoints, we have to adjust where we index. In this case, our rectangles are underestimates and we see we are off by an amount less than or equal to $\int_{n}^{\infty} f(x)dx$.



In conclusion, if R_N estimates our error after an Nth partial sum:

$$\int_{n+1}^{\infty} f(x)dx \leq R_N \leq \int_{n}^{\infty} f(x)dx$$

Warning: This was all done assuming that the hypotheses of the integral test were satisfied! If you want to use this remainder estimate, you need to know that the hypotheses of the integral test are satisfied!

37.2.1 Example

- Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 5 terms. Use the integral test remainder estimate to estimate the error involved in this approximation.
- How many terms are required to ensure the value of the sum is accurate to within 0.0005?

Solution:

- Plugging this into the calculator, we get:

$$\sum_{n=1}^{5} \frac{1}{n^3} \approx 1.185662$$

To use the integral test remainder, we first show that the three hypothese are satisfied:

  - $f(x) = \frac{1}{x^3}$ is continuous on $[1, \infty)$: the only VA is at $x = 0$
  - $f(x)$ is positive on $[1, \infty)$: since $x \geq 1$
  - $f(x)$ is decreasing on $[1, \infty)$: $f'(x) = -3x^{-4}$ is negative for all nonzero values of x.

Now we know the integral remainder test applies. Let's apply it! We did the $N = 5$th partial sum, so:

$$\int_6^{\infty} \frac{1}{x^3} dx \leq R_5 \leq \int_5^{\infty} \frac{1}{x^3} dx.$$

Evaluating these integrals, we get:

$$\frac{1}{72} \leq R_5 \leq \frac{1}{50}.$$

- To get $R_N \leq 0.0005$, we need

$$R_N \leq \int_N^{\infty} \frac{1}{x^4} dx \leq 0.0005$$

So find out what value of N makes the integral satisfy that inequality:

$$\begin{aligned} \int_N^{\infty} \frac{1}{x^3} dx &\leq 0.0005 \\ \lim_{T \to \infty} \int_N^T x^{-3} dx &\leq 0.0005 \\ \lim_{T \to \infty} \left(\frac{x^{-2}}{-2} \right) |_N^T &\leq 0.0005 \\ \frac{1}{2N^2} &\leq 0.0005 \\ 1 &\leq 0.001N^2 \\ 1000 &\leq N^2 \\ \sqrt{1000} \approx 31.62 &\leq N \end{aligned}$$

So taking $N = 32$ will make our partial sum solution within 0.0005 of the true solution.