AP Calculus BC
Review and Worksheet: Absolute Convergence and the Ratio and Root Tests

Given any series $\sum_{n=1}^{\infty} a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \cdots$$

whose terms are the absolute values of the trms of the original series.

A series $\Sigma\, a_n$ is called **absolutely convergent** is the series of absolute values $\Sigma\, |a_n|$ is convergent.

Quick note: If $\Sigma\, a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Example 1: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

is absolutely convergent because $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

is a convergent p-series (p = 2).

Example 2: We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent (meets the criteria of the Alternating Series Test), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with p = 1) and is therefore divergent.

A series $\Sigma\, a_n$ is called **conditionally convergent** if it is converegnt but not absolutely convergent (like the alternating harmonic series above).

If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

Example 3: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \;\cdot\;\cdot\;\cdot$$

is convergent or divergent.

The series has both positive and negative terms, but it is not alternating (The first term is positive, the next three are negative, and the following three are positive). We can apply the Direct Comparsion Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leq 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

We know that $\Sigma\, \frac{1}{n^2}$ is convergent (p-series with p = 2) and therefore $\Sigma\, |\cos n|$ is convergent by the Direct Comparison Test. Thus the given series $\Sigma\, \frac{(\cos n)}{n^2}$ is absolutely convergent and therefore convergent.

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \text{L} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent)

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \text{L} > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

Example 4: Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

We use the Ratio test with $a_n = (-1)^n \frac{n^3}{3^n}$:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^n (n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3} \left(\frac{n+1}{n}\right)^3 = \frac{1}{3} \left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

Problem 1: Use the Ratio Test to determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

Problem 2: Use the Ratio Test to determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$