Math 2300: Calculus **Spring 2019**

Lecture 35: Monday March 4

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35.1 8.4: Alternating Series

35.1.1 Alternating Sequence

Remember that sequences are lists, and series are what you get when you add it up.

When we first introduced sequences, we encountered a few different types of **alternating sequences:**

$$a_n = (-1)^n,\ a_n = \cos(n\pi), \text{ etc.}$$

We had an **alternating sequence convergence test**:

$$\text{If } \lim_{n\to\infty} |a_n| \to 0, \text{ then } \lim_{n\to\infty} a_n = 0.$$

We can use this to conclude that $a_n = \frac{(-1)^n}{n}$ converges. Now let's see if there are similar applications for series that sum up the terms of an alternating sequence:

35.1.2 Alternating Series

An alternating series will be something of the form:

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^2, \text{ or } \sum_{n=1}^{\infty}\frac{(-1)^n}{n}, \text{ or even } \sum_{n=1}^{\infty}\frac{\cos(n\pi)}{5\cdot 2^n}.$$

The identifying factor is that the terms of the series will alternate $+, -, +, -, \ldots$.

To test convergence of such a series, we have a new tool, called the **alternating series test**. This test has two hypotheses, and if these hypotheses are satisfied we can conclude convergence:

Let a_n be an alternating sequence. Then:

- if $|a_n|$ are decreasing and
- $\lim_{n\to\infty} |a_n| = 0$,

then the series $\sum_{n=1}^{\infty} a_n$ converges.

*Note: This is the dream test! All we have to do is show that the (absolute values of) the terms go to 0, and then the series converges! One might say it is *easier* for alternating series to converge than regular series. This makes sense: alternating series have a lot of cancellation inherently.

35.1.2.1 First Big Conclusion:

Remember the harmonic series?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Last week, we concluded this series diverged by using the integral comparison test. (We won't go through that again today.)

There is something called the **alternating harmonic series**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Let's see if this one converges or diverges... First, note that the absolute values of the terms are decreasing:

$$1 > \frac{1}{2} > \frac{1}{3} > \cdots.$$

We can also look at the limit of the $|a_n|$ as $n \to \infty$:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Since these two hypotheses are satisfied, we can conclude by the alternating series test that the series converges.

35.1.3 Recall Techniques to Show a Sequence is Decreasing:

If a_n is a sequence of positive numbers, we can show that it is decreasing by showing one of the following:

1. $a_{n+1} < a_n$
2. $\frac{a_{n+1}}{a_n} < 1$
3. $a_n - a_{n+1} > 0$
4. If we define a continuous differentiable function on $[1, \infty)$ such that $f(n) = a_n$, then a_n is decreasing iff $f'(x) \leq 0$ on $[1, \infty)$.

35.1.4 More Examples:

1. Determine if the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

converges or diverges.
Solution:
First, recognize that this is an alternating series, as $\cos(n\pi) = (-1)^n$ for $n = 1, 2, \ldots.$ Let's try to use the alternating series test:
First, show that the absolute values of the terms are decreasing:

$$|a_{n+1}| = \left| \frac{(-1)^{n+1}}{\sqrt{n+1}} \right| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = |a_n|,$$

so the sequence of terms of this series is decreasing in absolute value.
Now, investigate the limit:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

So we can conclude by the alternating series test that the series converges.

2. Determine if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

converges or diverges.
Solution:
First, let's consider the alternating series test. However, something is not quite right. Let's look at the limit first:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 1$$

Hmm! So we should suspect that this series might not converge. Let's try the divergence test. We need to show:

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} \neq 0.$$

As $n \to \infty$, this sequence alternates between approaching -1 and approaching 1, so it is certainly nonzero. Thus, we can conclude by the divergence test that this series diverges.

3. Determine if the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

converges or diverges.
Solution Outline:
Use the first derivative test to show that the sequence of absolute values of terms is decreasing.
Use l'Hopital's rule to show that the limit of the absolute values of the terms is 0.
$\Rightarrow$ conclude convergence by the alternating series test.