Math 2300: Calculus **Spring 2019**

Lecture 31: Tuesday February 26

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31.1 8.3: p Series and the Integral Test

31.2 The Integral Test

Big idea: Compare an infinite series to an integral to determine convergence/divergence.
If we define $f(x)$ such that $f(n) = a_n$, we can use the integral test when f is:

- **continuous** on $[1, \infty)$ (or $[k, \infty)$ if the series begins with index k)
- **positive** on $[1, \infty)$ (or $[k, \infty)$ if the series begins with index k)
- **decreasing** on $[1, \infty)$ (or $[k, \infty)$ if the series begins with index k).

When these hypotheses are satisfied, we can conclude:

- If $\int_1^\infty f(x)dx$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.
- If $\int_1^\infty f(x)dx$ diverges, then $\sum_{i=1}^{\infty} a_i$ diverges.

This is a powerful test, but note that it's important to make sure all hypotheses are satisfied before you try to use it! This means that, if you try to use it on an exam, you need to show that all of these hypotheses are satisfied.

31.2.1 Example:

Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

Our intuition looking at the numerator and denominator tells us we should probably expect convergence: e^n grows much faster than n. Let's use the integral test to prove it.

1. First, define $f(x) = \frac{x}{e^x}$.
2. $f(x)$ is continuous on $[1, \infty)$, since it is a function given by a polynomial over an exponential function, and so its denominator is nonzero.
3. $f(x)$ is positive for $x \geq 1$, since both numerator and denominator are ≥ 0.
4. To see that $f(x)$ is decreasing, let's take the derivative and show that it is always negative:

   $$f'(x) = \frac{e^x - e^x \cdot x}{e^{2x}}$$

   To see when this is negative, set up an inequality and consider it on the domain $[1, \infty)$:

   $$\begin{aligned} 0 &> \frac{e^x - e^x \cdot x}{e^{2x}} \\ 0 &> e^x(1 - x) \end{aligned}$$

   So $f'(x) = 0$ only when $x = 1$, and if we plug in anything greater than 1 we get a negative number: $f'(2) = \frac{e^2 - 2e^2}{e^4} = \frac{-1}{e^2} < 0$, so $f'(x)$ is negative on $[1, \infty)$ which means f is decreasing on $[1, \infty)$.
5. Now, we just determine whether or not the integral converges to make a decision about the series.

   $$\begin{aligned} \int_1^{\infty} f(x)dx &= \lim_{T\to\infty} \int_1^T \frac{x}{e^x} dx \\ &\text{Use integration by parts} \\ &\text{Let } u = x, dv = e^{-x}dx, \text{ so that} \\ &du = dx \text{ and } v = -e^{-x}: \\ &= \lim_{T\to\infty} \left(-xe^{-x}|_1^T + \int_1^T e^{-x} dx \right) \\ &= \lim_{T\to\infty} \left(-Te^{-T} + e^{-1} + -e^{-x}|_1^T \right) \\ &= \lim_{T\to\infty} \left(-Te^{-T} + e^{-1} + -e^{-T} + e^{-1} \right) \\ &= \frac{2}{e} + \lim_{T\to\infty} \frac{-(T+1)}{e^T} \\ &= \frac{\infty}{\infty} \text{ indeterminate form, so use l'Hopital's rule:} \\ &= \frac{2}{e} + \lim_{T\to\infty} \frac{-1}{e^T} \\ &= \frac{2}{e} \end{aligned}$$

   which is finite, so the integral converges. Thus, by the integral test, we can conclude that the series converges.

31.3 p Series Test

Let's use this to make conclusions about series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for $p > 0$.

1. First define $f(x) = \frac{1}{x^p}$
2. $f(x)$ is continuous on $[1, \infty)$, since it is a rational function with vertical asymptote at $x = 0$.
3. For $x \geq 1$, $f(x) > 0$, since numerator and denominator are both positive.
4. To see that $f(x)$ is decreasing, note that $\frac{1}{x^p} > \frac{1}{y^p}$ for all $y > x$.
5. Now, we can look at the integral and use the integral test to conclude convergence or divergence:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} dx &= \lim_{T\to\infty} \int_1^{T} \frac{1}{x^p} dx \\
&= \lim_{T\to\infty} \left(\frac{x^{-p+1}}{1-p} \right) \Big|_1^T \\
&= \lim_{T\to\infty} \frac{T^{-p+1}}{1-p} - \frac{1}{1-p} \\
&= \begin{cases} \frac{1}{p-1} & \text{, if } p > 1 \\ \infty & \text{, if } p \leq 1 \end{cases}
\end{aligned}$$

   So we can conclude by the integral p-test that this sum will converge for $p > 1$ and diverge for $p \leq 1$. We call this the (series) p-test.

31.3.1 Example

Determine whether the series is convergent or divergent:

$$1 + \frac{1}{4\sqrt[3]{2}} + \frac{1}{9\sqrt[3]{3}} + \frac{1}{16\sqrt[3]{4}} + \frac{1}{25\sqrt[3]{5}} + \cdots$$

Solution: First, find a way to write this as a series, so we can see what tools to use:

$$1 + \frac{1}{2^2\sqrt[3]{2}} + \frac{1}{3^2\sqrt[3]{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{7/3}}$$

We can use the p-test here! $p = 7/3 > 1$, so the series converges.

31.3.2 Example

Determine if $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ converges or diverges.
Solution:
It's not a p-test example, and it's not a geometric series. Let's see if we can use the integral test.

1. Let $f(x) = \frac{1}{x\ln(x)}$
2. $f(x)$ is continuous on $[2, \infty)$, since its domain is $x > 1$ and it has no discontinuities on that domain.
3. For $x \geq 2$, $f(x) > 0$, since $ln(x)$ and x are both positive for $x \geq 2$.
4. To see that $f(x)$ is decreasing, note that for $2 \leq x < y$, $\frac{1}{x} > \frac{1}{y}$ and $\frac{1}{\ln(x)} > \frac{1}{\ln y}$.
   Multiplying, we get $\frac{1}{x\ln x} > \frac{1}{y\ln y}$, so f is decreasing.
5. Now, we can use the integral test to make a conclusion:

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x\ln x} dx = \lim_{T\to\infty} \int_2^T \frac{1}{x\ln x} dx \\
&\text{Let } u = \ln(x), \text{ so } du = \frac{1}{x} dx, \\
&\text{when } x = 2,\ u = \ln 2, \text{ and } x = T,\ u = \ln T: \\
&= \lim_{T\to\infty} \int_{\ln 2}^{\ln T} u^{-1/2} du \\
&= \lim_{T\to\infty} 2u^{1/2}|_{\ln 2}^{\ln T} \\
&= \lim_{T\to\infty} 2T^{1/2} - 2 \cdot 2^{1/2} \\
&= \infty
\end{aligned}$$

So, since the integral $\int_2^{\infty} \frac{1}{x\ln x} dx$ diverges, we conclude that the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ diverges by the integral test.