Math 2300: Calculus **Spring 2019**

Lecture 25: Monday February 18

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WebAssign due tonight

25.1 Sequences

Think of a **sequence** as a list of numbers:

$$a_1, a_2, a_3, ..., a_k, ...$$

But really it is a function whose domain is positive integers:

$$f(1) = a_1,\ f(2) = a_2,\ ...,\ f(k) = a_k, ...$$

We are often interested in the end behavior of a sequences, $\lim_{n\to\infty} a_n$. In terms of the function picture of a sequence:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

This is helpful, because we have Calc 1 tools to help us with limits.
subsectionExample: Consider the sequence:

$$1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{k^2}, \ldots$$

So $f(x) = \frac{1}{x^2}$ is the function that gives us this sequence, when we take the domain to be positive integers. What is the end behavior of this sequence?

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

25.1.1 Tools To Remember

1. Squeeze Law (Sandwich Theorem): If $f(x) \leq h(x) \leq g(x)$ for all x and

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = C,$$

   then $\lim_{x\to\infty} h(x) = C$, as well.

2. Showing an alternating sequence converges:
   If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

3. Showing an alternating sequence diverges:

4. Leading Coefficient Test from limits of rational functions:
   If $f(x)$ and $g(x)$ are polynomials with the same degree, then $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$ = the ratio of their leading

coefficients. To **prove** this:

$$\begin{aligned}\lim_{n\to\infty}\frac{n^3+2n+1}{2n^3-1} &= \lim_{n\to\infty}\frac{n^3+2n+1}{2n^3-1}\cdot\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\\ &= \lim_{n\to\infty}\frac{\frac{n^3}{n^3}+\frac{2n}{n^3}+\frac{1}{n^3}}{\frac{2n^3}{n^3}-\frac{1}{n^3}}\\ &= \lim_{n\to\infty}\frac{1+\frac{2}{n^2}+\frac{1}{n^3}}{2-\frac{1}{n^3}}\\ &= \frac{1}{2}\end{aligned}$$

25.1.2 Three ways to show a sequence is decreasing:

1. If $a_n > a_{n+1}$ for all n, then the sequence is decreasing.

2. If $a_n - a_{n+1} > 0$ for all n, then the sequence is decreasing.

3. If $\frac{a_{n+1}}{a_n} < 1$ for all n, then the sequence is decreasing.

25.1.3 Some Vocab

A sequence is **bounded** if there exists some number L such that $|a_n| \leq L$ for all L.
A sequence is **monotonic** if it is either strictly increasing for all x or strictly decreasing for all x.
Bounded monotonic sequences must converge. (Think about why?)

A **recursively** defined sequence is one where the definition of one term depends on the previous ones. You may be familiar with the Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, ...$ where $f_n = f_{n-1} + f_{n-2}$.

25.1.4 Some Fun Examples

25.1.4.1 Example 1:

Write an expression for a_n for the sequence that begins

$$\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \ldots$$

Solution:
Strategy: Find a way of expressing the numerator in terms of what term it is (n), and do the same for the denominator.
$a_n = \frac{2n}{3^n}$.

25.1.4.2 Example 2:

Write an expression for a_n for the sequence that begins

$$7, \frac{-9}{2}, \frac{11}{6}, \frac{-13}{24}, \ldots$$

Solution:
Same strategy as before.
Numerator: $7, -9, 11, -13, ...$ seems to be $(-1)^{n+1}(2n+5)$
Denominator: $1, 2, 6, 24, ...$ let's try to see the factors of this: $1, 2, 6 = 2 \cdot 3, 24 = 4 \cdot 3 \cdot 2, ...$ Seems to be $n!$.
$a_n = \frac{(-1)^{n+1}(2n+5)}{n!}$

25.1.5 Some Limit Examples

25.1.5.1 Example 1

Suppose $a_n = \frac{(-1)^n \ln(n)}{n}$. Find $\lim\limits_{n\to\infty} a_n$.
Solution:
Here we can use L'Hopital's rule. Let $f(x)$ be defined $f(x) = \frac{(-1)^x \ln(x)}{x}$ so that $f(n) = a_n$ for each integer $n = 1, 2, ...$ *We need to do this in order to use L'H, since L'H was only defined on differentiable functions.* We won't always need to do this, but we do because L'H requires differentiable functions.
First, notice that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$, so we can just check the absolute value:

$$\lim_{n\to\infty} |a_n| = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

Since the limit goes to ∞/∞, we can apply L'H:

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$
$$= \lim_{x\to\infty} \frac{1}{x}$$
$$= 0$$

Since $\lim\limits_{n\to\infty} |a_n| = 0$, we conclude $\lim\limits_{n\to\infty} a_n = 0$.

25.1.5.2 Example 2

Suppose $a_n = \frac{\sqrt{3n^2+4}}{n-1}$. Find $\lim\limits_{n\to\infty} a_n$.
Solution:
With this one, we can use our "multiplying numerator and denominator by a factor to cancel high powers" trick.

$$\lim_{n\to\infty} \frac{\sqrt{3n^2+4}}{n-1} = \lim_{n\to\infty} \frac{\sqrt{3n^2+4}}{n-1} \cdot \frac{\frac{1}{\sqrt{n^2}}}{\frac{1}{n}}$$

Note the trick we have to do to get the $1/n$ under the radical!

$$= \lim_{n\to\infty} \frac{\sqrt{3+\frac{4}{n^2}}}{1-\frac{1}{n}}$$

$$= \sqrt{3}$$

25.1.5.3 Example 3

Suppose $a_n = \left(1+\frac{1}{n}\right)^n$. Find $\lim\limits_{n\to\infty} a_n$.
Solution:
This one has a trick to it, but it should be familiar...Let's use $f(x) = \left(1+\frac{1}{x}\right)^x$, so that we can use L'H if necessary.
Let's call this limit L, and then we can make use of log's. Note that I write log for ln.:

$$L = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$

log is a continuous function, so we can bring it inside the limit:

$$\log(L) = \lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right)$$

Letting $x \to \infty$ is equivalent to letting $1/x \to 0$:
Let $s = 1/x$:

$$\log(L) = \lim_{s\to 0} \frac{\log(1+s)}{s}$$

This is $0/0$, so we can apply L'H:

$$\log(L) = \lim_{s\to 0} \frac{1/(1+s)}{1}$$

$$\log(L) = 1$$

$$L = e$$

So we conclude $\lim\limits_{n\to\infty} a_n = e$.

25.1.6 Some Increasing/Decreasing Examples

25.1.6.1 Example 1

Show $a_n = \frac{3^{n+2}}{5^n}$ is decreasing.
Solution:

Look at the ratio $\frac{a_{n+1}}{a_n}$:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{3^{n+3}/5^{n+1}}{3^{n+2}/5^n} \\ &= \frac{3}{5}\end{aligned}$$

Since $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 1$, it follows that $a_{n+1} < a_n$ for all $n \geq 1$. Hence, the sequence is decreasing.

25.1.6.2 Example 2

Show $a_n = \frac{n}{2^n}$ is decreasing.
Solution:
Let's subtract this time, since we don't have all exponentials:

$$\begin{aligned}a_n - a_{n+1} &= \frac{n}{2^n} - \frac{n+1}{2^{n+1}} \\ &= \frac{2n - (n+1)}{2^{n+1}} \\ &= \frac{n-1}{2^{n+1}}\end{aligned}$$

For $n = 1$, this is 0, so $a_1 = a_2$. For $n > 1$, $\frac{n-1}{2^{n+1}} > 0$, so we have $a_n - a_{n+1} > 0$ and thus $a_n > a_{n+1}$. So this sequence is only monotonically decreasing for $n > 1$.

25.1.6.3 Example 3

Show $a_n = \frac{n}{n+1}$ is increasing.
Solution:
This one is similar to the last one:

$$\begin{aligned}a_n - a_{n+1} &= \frac{n}{n+1} - \frac{n+1}{n+2} \\ &= \frac{n(n+2)}{(n+1)(n+2)} - \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n^2 + 2n - (n^2 + 2n + 1)}{(n+1)(n+2)} \\ &= \frac{-1}{(n+1)(n+2)}\end{aligned}$$

This is negative for all $n \geq 1$, so $a_n - a_{n+1} < 0$ and thus $a_n < a_{n+1}$, which shows the sequence is increasing.

25.1.7 Some Recursion Examples

Not the most important thing to emphasize, but it'll make your life easier if you can get the hang of this too. **General strategy:** Write out a few terms and look for the pattern. Look for the dependence on what term it is.

25.1.7.1 Example 1

Find a formula for a_n if $a_1 = 2$ and $a_{n+1} = a_n + 5$.
Solution:
The first few terms are:

$$2, 7, 12, 17, 22, ...$$

So $a_n = 5(n-1) + 2 = 5n - 3$.
This type of sequence is called **arithmetic**, because we just keep adding 5 to get to the next term. Whenever you just add (or subtract...which is adding a negative), it's an arithmetic sequence.

25.1.7.2 Example 2

Find a formula for a_n if $a_1 = 4$ and $a_{n+1} = 5a_n$.
Solution:
The first few terms are:

$$4, 20, 100, 500, ...$$

So $a_n = 4 \cdot 5^{n-1}$.
This type of sequence is called **geometric**, because we just keep multiplying by 5 to get to the next term. Whenever you just multiply (or divide...which is multiplying by a fraction), it's an arithmetic sequence.