Math 2300: Calculus **Spring 2019**

Lecture 24: Friday February 15

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WebAssign due tonight

24.1 Moments and Center of Mass

24.1.1 Center of Mass

The **center of mass** (or centroid, or center of moments) is the balance point of an object (a ”flat” 2-dimensional object, or a 1-dimentional object. It gets more complicated with volumes...).

For a system of n particles with masses $m_1, m_2, ..., m_n$ located at the points $(x_1, y_1), ..., (x_n, y_n)$ in the plane, the **center of mass of the system** is located at $(\overline{x}, \overline{y})$, where:

$$\overline{x} = \frac{x_1 m_1 + \cdots + x_n m_n}{m_1 + \cdots + m_n}$$

$$\overline{y} = \frac{y_1 m_1 + \cdots + y_n m_n}{m_1 + \cdots + m_n}$$

This should be pretty intuitive: Think of balancing a rectangular piece of wood on your finger:

- Where would you place your finger to balance it?
- What if the mass of one side was heavier than the other: Then where would you place your finger?

24.1.1.1 Examples:

1. Find the center of mass of the system of the following point masses:
   - A mass of 5 at (1,4)
   - A mass of 2 at (3,-2)
   - A mass of 10 at (-1,-4)

   Solution:
   Begin by sketching the points in the plane.
   Using the equations:

   $$\overline{x} = \frac{1 \cdot 5 + 3 \cdot 2 + (-1) \cdot 10}{5 + 2 + 10} = \frac{1}{17}$$

   $$\overline{y} = \frac{4 \cdot 5 + (-2) \cdot 2 + (-4) \cdot 10}{5 + 2 + 10} = \frac{-24}{17}$$

   So the center of mass is located at $(1/17, -24/17)$.

24.1.2 Moments

The **moment of the system about the y-axis**, denoted M_y, is the tendency of the system to rotate about the y-axis.
The **moment of the system about the x-axis**, denoted M_x, is the tendency of the system to rotate about the x-axis. Their equations are:

$$M_y = x_1 m_1 + \cdots + x_n m_n = \sum_{i=1}^{n} x_i m_i$$

$$M_x = y_1 m_1 + \cdots + y_n m_n = \sum_{i=1}^{n} y_i m_i$$

(Show moment stuff on website: `http://web.mit.edu/4.441/1_lectures/1_lecture5/1_lecture5.html`)

24.1.3 Putting it all together

Notice that M_x and M_y appear in our equations for the COM:

$$\overline{x} = \frac{M_y}{\text{mass}}$$

$$\overline{y} = \frac{M_x}{\text{mass}}$$

What if we have a thin region bounded by curves and we want to find the center of mass? Say $f(x)$, $g(x)$ bounds a region between $x = a$ and $x = b$, with $f(x) \geq g(x)$ on $[a, b]$. (Sketch).
Then: take a small slice. What is the center of mass of this slice?

$$\widetilde{x} = x$$

$$\widetilde{y} = \frac{f(x) + g(x)}{2}$$

What is the mass of this slice?

$$\text{mass} = \text{Area} \cdot (\text{Mass density of region}) = [f(x) - g(x)] \cdot dx \cdot \rho$$

Putting this together with out knowledge that infinite sums are just integrals, we get:

$$\overline{x} = \frac{M_x}{\text{total mass}} = \frac{\int_a^b \widetilde{x} \cdot (\text{mass})}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\rho \int_a^b x(f(x) - g(x))dx}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

$$\overline{y} = \frac{M_x}{\text{mass}} = \frac{\int_a^b \widetilde{y} \cdot (\text{mass})}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\rho \int_a^b \frac{f(x)+g(x)}{2} (f(x) - g(x))dx}{\rho \int_a^b (f(x) - g(x))dx} = \frac{\frac{1}{2} \int_a^b (f(x) + g(x))(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

We can simplify this a bit as:

$$\overline{y} = \frac{\int_a^b (f^2(x) - g^2(x))dx}{2 \int_a^b (f(x) - g(x))dx}$$

This gives us a way to compute centers of mass of bounded regions between curves. Let's use it! But before that, what's with ρ cancelling out? Since we are assuming the density is uniform across the object, it basically doesn't matter what it is! The center of mass will be the same if it's really dense vs. not dense, because the COM is just a balancing point.

24.1.4 Example 1:

Find the centroid of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.
Solution:
(Sketch) In this example, $f(x) = \sqrt{x}$ and $g(x) = x$, and the intersect between 0 and 1. Putting this information into our formulas:

$$\overline{x} = \frac{\int_0^1 x(\sqrt{x} - x)dx}{\int_0^1 (\sqrt{x} - x)dx} = \frac{\int_0^1 (x^{3/2} - x^2)dx}{\int_0^1 (x^{1/2} - x)dx} = \frac{[(2/5)(x^{5/2}) - (1/3)x^3]|_0^1}{[(2/3)x^{3/2} - (1/2)x^2]|_0^1} = \frac{(2/5) - (1/3)}{(2/3) - (1/2)} = \frac{1/15}{1/6} = \frac{6}{15} = \frac{2}{5}$$

$$\overline{y} = \frac{\int_0^1 (x - x^2)dx}{2\int_0^1 (x^{1/2} - x)dx} = \frac{[(1/2)x^2 - (1/3)x^3]|_0^1}{2[(2/3)x^{3/2} - (1/2)x^2]|_0^1} = \frac{1/6}{2(1/6)} = \frac{1}{2}$$

So the center of mass is $(2/5, 1/2)$.

24.1.5 Example 2:

Find the center of mass of a semicircular plate of radius r.
Solution:
(Sketch) In this example, $f(x) = \sqrt{r^2 - x^2}$ and $g(x) = 0$ (the x-axis), and they intersect between $x = -r$ and $x = r$.

$$\overline{x} = \frac{\int_{-r}^r x\sqrt{r^2 - x^2}dx}{\int_{-r}^r \sqrt{r^2 - x^2}dx}$$

$$\overline{y} = \frac{\int_{-r}^r (r^2 - x^2)dx}{2\int_{-r}^r \sqrt{r^2 - x^2}dx}$$

Integrate to find the solution: COM $= (0, 4r/(3\pi))$.