Math 2300: Calculus **Spring 2019**

Lecture 15: Monday February 4

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WebAssign due tonight

15.1 Areas of Regions and Volumes of Solids of Revolution

The goal of today is to review ways of finding geometric areas, and to extend that notion to volumes.

15.1.1 Example 1:

Find the area bounded between $x = 2y^2$ and $x = 27 - y^2$.
Step 1: Draw a picture! Find where these two graphs intersect:

$$\begin{aligned} 2y^2 &= 27 - y^2 \\ 3y^2 - 27 &= 0 \\ 3(y+3)(y-3) &= 0 \end{aligned}$$

So the points of intersection occur when $y = \pm 3$: $(18, -3), (18, 3)$.
Now we can sketch the region (sketch the graphs)
Step 2: Write an integral describing the area of the region you've sketched. Here, the $x = 27 - y^2$ graph is always to the right of the $x = 2y^2$ graph, so our area can be written:

$$\int_{-3}^{3} (27 - y^2 - (2y^2))dy$$

Step 3: Integrate:

$$\begin{aligned} \int_{-3}^{3} (27 - y^2 - (2y^2))dy &= \int_{-3}^{3} (27 - 3y^2)dy \\ &= (27y - y^3)|_{-3}^{3} \\ &= (81 - 27) - (-81 + 27) \\ &= 108 \end{aligned}$$

Notice: One graph was to the **right** of the other, so we used a dy integral.

15.1.2 Example 2:

Find the area bounded between $y = 5x - x^2$ and $y = x$.
Step 1: Draw a picture! Find where these two graphs intersect in order to draw an accurate, useful picture:

$$\begin{aligned} 5x - x^2 &= x \\ 4x - x^2 &= 0 \\ x(4 - x) &= 0 \end{aligned}$$

So the points of intersection are at $x = 0$, $x = 4$: namely, $(0, 0), (4, 4)$.
Now we can sketch the region (sketch graphs).
Step 2: Write an integral to describe the area of the region you've just sketched.
Here, the parabola is always above the line, so our area can be written:

$$\int_0^4 (5x - x^2 - (x))dx$$

Step 3: Integrate:

$$\begin{aligned}\int_0^4 (5x - x^2 - (x))dx &= \int_0^4 (4x - x^2)dx \\ &= 2x^2 - \frac{x^3}{3}|_0^4 \\ &= 32 - \frac{64}{3} \\ &= \frac{32}{3}\end{aligned}$$

Notice: One graph was **above** the other, so we used a dx integral.

15.1.3 Summary

It's either

$$\int_{x=a}^{x=b} (top - bottom)dx$$

or

$$\int_{y=a}^{y=b} (right - left)dy.$$

15.2 Volumes

So we've covered 2-d...what if we want to use integrals to calculate volumes? Let's use our last example. Imagine rotating this region around the x-axis to create some kind of container. If we take cross sections of this shape, the cross sections will look like **washers**.
How can we describe this washer shape? What is the inner radius? The outer radius? Draw it.
The inner radius r_1 of this washer comes from the function $y = x$, so $r_1 = x$; the outer radius comes from $y = 5x - x^2$, so $r_2 = 5x - x^2$.
Find the area of one washer-shaped slice: The area of a circle is πr^2, so the area of a washer is $\pi r_2^2 - \pi r_1^2$. In our case, the area is:

$$\pi(5x - x^2)^2 - \pi x^2 = \pi(25x^2 - 10x^3 + x^4) - \pi x^2 = \pi(24x^2 - 10x^3 + x^4)$$

Let's integrate this area across the x-axis, from $x = 0$ to $x = 4$, encompassing the whole volume of our 3d solid:

$$\begin{aligned}\int_0^4 \pi(24x^2 - 10x^3 + x^4)dx &= \pi \int_0^4 (24x^2 - 10x^3 + x^4)dx \\ &= \pi(8x^3 - \frac{5}{2}x^4 + \frac{x^5}{5})|_0^4 \\ &= \pi \cdot (8 \cdot 64 - \frac{5 \cdot 256}{2} + \frac{4^5}{5}) \\ &= 76.8\pi\end{aligned}$$

15.2.1 Example:

Find the volume of the solid V obtained by rotating the region bounded by the following curves around the x-axis:

$$y = 4 - \frac{1}{2}x,\ y = 0,\ x = 1,\ x = 4$$

Start with a sketch! If you can't sketch the region, you'll have extreme difficulty finding out how to write the appropriate integral.
Start by finding where the curves intersect:

$$y = 4 - \frac{1}{2}(1) \Rightarrow y = \frac{7}{2} \Rightarrow (1, \frac{7}{2})$$

$$y = 4 - \frac{1}{2}(4) \Rightarrow y = 2 \Rightarrow (4, 2)$$

Sketch the area to be rotated.
Find the area of one slice of washer: This washer will actually just be a disc, with area $\pi r^2 = \pi(4 - \frac{1}{2}x)^2$.
Integrate this area across the region:

$$\begin{aligned}&\int_1^4 \pi(4 - \frac{1}{2}x)^2 dx \\ &= \pi \int_1^4 (16 - 4x + \frac{1}{4}x^2)dx \\ &= \pi(16x - 2x^2 + \frac{1}{12}x^3)|_1^4 \\ &= \pi(64 - 32 + \frac{64}{12} - (16 - 2 + \frac{1}{12})) \\ &= \pi(32 + \frac{64}{12} - \frac{1}{12} - 14) \\ &= \pi(18 + \frac{21}{4}) \\ &= \frac{93\pi}{4}\end{aligned}$$

This is what we will call **washer method**.

15.2.2 Another Example:

Rotate the region bounded by $y = \sqrt{x}, y = 3$ and the y-axis around the y-axis.
Hint: In this case, our washers are going to be horizontal.

Sketch the region.
Notice that the washers have radii that are x-values on the curve $y = \sqrt{x}$. We will need to express these as functions of y, so that we can integrate up and down.
Radius = yˆ2

$$\int_0^3 \pi (y^2)^2 dy$$

$$= \pi \int_0^2 y^4 dy$$

$$= \pi (\frac{y^5}{5})|_0^3$$

$$= \frac{3^5 \pi}{5}$$