Math 2300: Calculus **Spring 2019**

Lecture 14: Friday February 1

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WebAssign due tonight

14.1 Warm-up

Evaluate the following integral:

$$\int_1^2 \frac{1}{\sqrt{x-1}}dx$$

$$\begin{aligned}
\int_1^2 \frac{1}{\sqrt{x-1}}dx &= \lim_{T\to 1^+}\int_T^2 \frac{1}{\sqrt{x-1}}dx \\
&= \lim_{T\to 1^+}\int_T^2 (x-1)^{-1/2}dx \\
&= \lim_{T\to 1^+} 2(x-1)^{1/2}|_T^2 \\
&= \lim_{T\to 1^+} 2(1)^{1/2} - 2(T-1)^{1/2} \\
&= 2 - 2\lim_{T\to 1^+}(T-1)^{1/2} \\
&= 2
\end{aligned}$$

14.2 Improper Integrals by Comparison

Sometimes, even with all of the techniques we have, we may not want to evaluate an integral. We may see that it diverges, but proving it directly would be difficult. For example:
We know $\int_1^\infty \frac{1}{x}dx$ diverges. What about $\int_1^\infty \frac{\sin(x)+5}{x}dx$? Intuitively, we know that the graph of $\frac{\sin(x)+5}{x}$ is going to be even higher above the x-axis than $\frac{1}{x}$, since $\sin(x)+5$ is larger than 1.
Stating this mathematically as a **hypothesis:**
On the interval $[1,\infty)$, $0 \le \frac{1}{x} \le \frac{\sin(x)+5}{x}$.
Conclusion: Integrating this inequality, we get:

$$0 \le \int_1^\infty \frac{1}{x}dx \le \int_1^\infty \frac{\sin(x)+5}{x}dx.$$

Since the middle integral diverges, the larger interval must diverge as well.
This is how we show $\int_1^\infty \frac{\sin(x)+5}{x}dx$ diverges.
This is called the Integral Comparison Test. We can use this to decide whether an integral converges or diverges, but not to get the value of the integral in the case it converges.
Tip 1: Note that the functions need to be above the x-axis on the interval of integration. Negative areas don't work well with this test.
Tip 2: Choose a good function to compare it with. Often comparing it using the p-test is helpful. But really just choose any integral that you can show converges/diverges (depending on your conclusion).

14.2.1 Example:

Use the comparison test to determine whether the following integral converges or diverges.

$$\int_1^\infty \frac{|\sin(x)|}{x^2+1}dx$$

First, note that we can integrate $\int_1^\infty \frac{1}{x^2+1}dx$:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2+1}dx &= \lim_{T\to\infty} \int_1^T \frac{1}{x^2+1}dx \\
&= \lim_{T\to\infty} \arctan(x)|_1^T \\
&= \lim_{T\to\infty} \arctan(T) - \arctan(1) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4} < \infty
\end{aligned}$$

Now, compare the integrands:

$$\text{Since } 0 \leq \frac{|\sin(x)|}{x^2+1} \leq \frac{1}{x^2+1} \text{ on } [1,\infty)$$

We can integrate to conclude:

$$0 \leq \int_1^\infty \frac{|\sin(x)|}{x^2+1}dx \leq \int_1^\infty \frac{1}{x^2+1}dx.$$

Since the integral on the right (the largest integral) is finite, the middle one must be finite too. Thus, we conclude $\int_1^\infty \frac{|\sin(x)|}{x^2+1}dx$ converges.

14.2.2 Example:

Use the comparison test to determine whether the following integral converges or diverges.

$$\int_3^\infty \frac{\ln(x)}{\sqrt{x}}dx$$

For this one, we can compare with $\int_3^\infty \frac{1}{\sqrt{x}}dx$. This integral diverges by the p-test, since $p = 1/2 < 1$. Now, we can write the hypothesis:

$$\text{Since } 0 \leq \frac{1}{\sqrt{x}} \leq \frac{\ln(x)}{\sqrt{x}}$$

We can integrate to conclude:

$$0 \leq \int_3^\infty \frac{1}{\sqrt{x}}dx \leq \int_3^\infty \frac{\ln(x)}{\sqrt{x}}dx$$

Since the integral in the middle diverges, the integral on the right (the largest integral) must diverge as well. Thus we conclude $\int_3^\infty \frac{\ln(x)}{\sqrt{x}}dx$ diverges.