Math 2300: Calculus **Fall 2019**

Lecture 12:

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12.1 Improper Integrals with Limits

Sometimes, it's possible to express an infinite area with the definite integral symbol:

$$\int_0^\infty 1dx$$

If we visualize this integral, it represents the area under the curve $y = 1$ to the right of the x-value $x = 0$. This area is very clearly infinite. But not all infinite integrals are easy to evaluate. What about

$$\int_1^\infty \frac{1}{x}dx?$$

or

$$\int_3^\infty \frac{1}{9+x^2}dx$$

Some integrals express areas that would take infinite space to draw on paper, the area itself is actually a finite number. It's hard to tell when areas will be infinite and when they will be finite - we'll learn how to do it using limits!

12.1.1 Integrals Over Infinite Intervals

The way to approach this is to use limits. Rewrite:

$$\int_1^\infty \frac{1}{x}dx = \lim_{T\to\infty}\int_1^T \frac{1}{x}dx$$

We can evaluate the integral now:

$$= \lim_{T\to\infty} \ln|x|\big|_1^T$$

$$= \lim_{T\to\infty} \ln|T| - \ln|1|$$

$$= \lim_{T\to\infty} \ln|T|$$

Now that we've gotten rid of the integral, we can see clearly that the limit diverges.

12.1.1.1 Example 2:

$$\int_3^\infty \frac{1}{9+x^2}dx$$

$$\begin{aligned}
\int_3^\infty \frac{1}{9+x^2}dx &= \lim_{T\to\infty}\int_3^T \frac{1}{9+x^2}dx \\
&= \lim_{T\to\infty}\frac{1}{9}\int_3^T \frac{1}{1+\left(\frac{x}{3}\right)^2}dx \\
&= \lim_{T\to\infty}\frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_3^T \\
&= \frac{1}{3}\lim_{T\to\infty}\arctan(T/3)-\arctan(1) \\
&= \frac{1}{3}(\pi/2-\pi/4) \\
&= \frac{\pi}{12}
\end{aligned}$$

So this one converges (as long as we remember what the graph of $y=\arctan(x)$ looks like...!)

12.1.1.2 Example 3:

$$\int_1^\infty \frac{1}{x^p}dx, \text{ where } p\neq 1$$

We are doing this example in hopes of formulating a rule for this situation

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}dx &= \int_1^\infty x^{-p}dx \\
&= \lim_{T\to\infty}\int_1^T x^{-p}dx \\
&= \lim_{T\to\infty}\frac{x^{-p+1}}{-p+1}\Big|_1^T \\
&= \lim_{T\to\infty}\frac{T^{-p+1}}{-p+1}-\frac{1^{-p+1}}{-p+1}
\end{aligned}$$

Notice that it doesn't really matter what positive value we put as the lower bound - that part always converges. Negative values would cross a vertical asymptote...we'll handle this case next.

To see when $\lim_{T\to\infty}\frac{T^{-p+1}}{-p+1}$ is finite, it might be more clear to re-write:

$$\lim_{T\to\infty}\frac{T^{-p+1}}{-p+1}=\lim_{T\to\infty}\frac{1}{(-p+1)T^{p-1}}$$

When $p - 1 > 0$, this will converge. When $p - 1 < 0$, this will diverge.
Try some values of p to confirm.
This is the **p-test**:

$$\int_1^\infty \frac{1}{x^p} dx \to \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

12.1.2 Integrals Over Vertical Asymptotes

It is **your job** to know whether or not you are integrating on an interval that includes a vertical asymptote of the integrand. To integrate over an asymptote, you must break up the interval at the asymptote and use a limit.

12.1.2.1 Example 1:

$$\int_0^1 \ln(x) dx$$

We know that $y = \ln(x)$ has a vertical asymptote at $x = 0$. So we must re-write this integral using a limit:

$$\int_0^1 \ln(x)dx = \lim_{T\to 0^+} \int_T^1 \ln(x)dx$$

Now use integration by parts: $u = ln(x)$, $du = \frac{1}{x}dx$, $dv = dx$, $v = x$:

$$= \lim_{T\to 0^+} \left[\ln(x)x|_T^1 - \int_T^1 1dx \right]$$
$$= \lim_{T\to 0^+} (\ln(1)\cdot 1 - \ln(T)\cdot T - (1-T))$$
$$= \lim_{T\to 0^+} (-T\ln(T) - 1 + T)$$
$$= \lim_{T\to 0^+} (-T\ln(T)) - \lim_{T\to 0^+}(1) + \lim_{T\to 0^+}(T)$$
$$= \lim_{T\to 0^+} (-T\ln(T)) - 1 + 0$$

Use l'Hopital's rule to evaluate this limit:

$$= -1 + \lim_{T\to 0^+} \frac{\ln(T)}{1/T}$$
$$= -1 + \lim_{T\to 0^+} \frac{1/T}{-1/T^2}$$
$$= -1 + \lim_{T\to 0^+} \frac{1}{-1/T}$$
$$= -1 + 0$$
$$= -1$$

Make sure you remember l'Hopital's rule. It's going to come up a lot!

12.1.2.2 Example 2:

$$\int_1^4 \frac{1}{x-2}dx$$

Note that this one has a vertical asymptote at $x = 2$, and two is contained in our interval of integration: $[1, 4]$. We will need to split this integral up and use limits:

$$\begin{aligned}
\int_1^4 \frac{1}{x-2}dx &= \int_1^2 \frac{1}{x-2}dx + \int_2^4 \frac{1}{x-2}dx \\
&= \lim_{T\to 2^-} \int_1^T \frac{1}{x-2}dx + \lim_{S\to 2^+} \int_S^4 \frac{1}{x-2}dx \\
&= \lim_{T\to 2^-} \ln(|x-2|)\big|_1^T + \lim_{S\to 2^+} \ln(|x-2|)\big|_S^4 \\
&= \lim_{T\to 2^-} (\ln(|T-2|) - \ln(1)) + \lim_{S\to 2^+} (\ln(2) - \ln(|S-2|)) \\
&= \lim_{T\to 2^-} \ln(|T-2|) + \lim_{S\to 2^+} (-\ln(|S-2|))
\end{aligned}$$

This one diverges, because the individual limits diverge to ∞ and $-\infty$. Even if only one of them diverged, we would say that this limit diverges.

12.1.3 Miscellaneous

12.1.3.1 What if you have two infinite bounds?

If you have an integral of the form $\int_{-\infty}^{\infty}$, you need to break it into two integrals: $\int_{-\infty}^{0} + \int_0^{\infty}$ and **then** use limits. You can use any point - it doesn't have to be 0, but 0 is often convenient.
Example:

$$\int_{-\infty}^{\infty} te^{-t^2}dt$$

$$\begin{aligned}
\int_{-\infty}^{\infty} te^{-t^2}dt &= \int_{-\infty}^{0} te^{-t^2}dt + \int_0^{\infty} te^{-t^2}dt \\
&= \lim_{T\to-\infty} \int_T^0 te^{-t^2}dt + \lim_{S\to\infty} \int_0^S te^{-t^2}dt
\end{aligned}$$

Use u substitution with $u = -t^2$, so $\frac{-1}{2}du = tdt$. Note that the bounds flip sign:

$$\begin{aligned}
&= \lim_{T\to-\infty} \frac{-1}{2} \int_{-T^2}^0 e^u du + \lim_{S\to\infty} \frac{-1}{2} \int_0^{-S^2} e^u du \\
&= \lim_{T\to-\infty} \frac{-1}{2} e^u\big|_{-T^2}^0 + \lim_{S\to\infty} \frac{-1}{2} e^u\big|_0^{-S^2} \\
&= \lim_{T\to-\infty} \frac{-1}{2}(e^0 - e^{-T^2}) + \lim_{S\to\infty} \frac{-1}{2}(e^{-S^2} - e^0) \\
&= \frac{-1}{2}(1-0) + \frac{-1}{2}(0-1) \\
&= 0
\end{aligned}$$