Math 2300.013: Calculus — Spring 2019

Lecture 10: Practice With Integrals

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Evaluate the following integrals.

1. $\int \cot(10x)\csc^4(10x)dx$

since $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

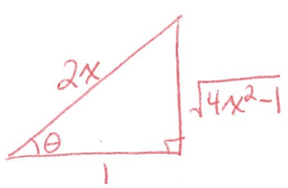
$$= \int \csc^3(10x)\cdot\csc(10x)\cot(10x)\,dx$$

let $u = \csc(10x) \rightarrow du = -10\csc(10x)\cot(10x)dx$

$-\frac{1}{10}du = \csc(10x)\cot(10x)dx$

$$= -\frac{1}{10}\int u^3 du = \frac{-1}{10}\left(\frac{u^4}{4}\right) + C = \boxed{\frac{-1}{40}\csc^4(10x) + C}$$

2. $\int x^3(4x^2-1)^{5/2}dx = \int x^3\left(\sqrt{4x^2-1}\right)^5 dx$



$\sec\theta = 2x$ $\quad$ $\tan\theta = \sqrt{4x^2-1}$

$\frac{1}{2}\sec\theta = x$

$dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$= \int\left(\tfrac{1}{2}\sec\theta\right)^3(\tan\theta)^5\cdot\tfrac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{16}\int \sec^4\theta\tan^6\theta\,d\theta$$

since $\frac{d}{d\theta}\tan\theta = \sec^2\theta\ldots$

$$= \frac{1}{16}\int(1+\tan^2\theta)\tan^6\theta\sec^2\theta\,d\theta$$

let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$

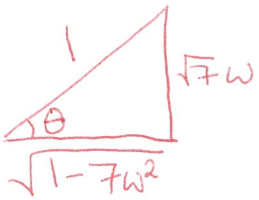
$$= \frac{1}{16}\int(1+u^2)u^6\,du$$

$$= \frac{1}{16}\int(u^6+u^8)du$$

$$= \frac{1}{16}\left(\frac{u^7}{7}+\frac{u^9}{9}\right)+C = \frac{1}{16}\left(\frac{\tan^7\theta}{7}+\frac{\tan^9\theta}{9}\right)+C$$

$$= \boxed{\frac{1}{16}\left(\frac{\left(\sqrt{4x^2-1}\right)^7}{7}+\frac{\left(\sqrt{4x^2-1}\right)^9}{9}\right)+C}$$

3. $\int \sqrt{1-7w^2}\,dw$



$\sin\theta = \sqrt{7}w$

$\frac{1}{\sqrt{7}}\sin\theta = w$

$dw = \frac{1}{\sqrt{7}}\cos\theta\, d\theta$

$\cos(\theta) = \sqrt{1-7w^2}$

$$= \int \cos\theta \cdot \frac{1}{\sqrt{7}}\cos\theta\, d\theta$$

$$= \frac{1}{\sqrt{7}}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{\sqrt{7}}\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{2\sqrt{7}}\int (1+\cos(2\theta))\, d\theta$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \frac{1}{2\sqrt{7}}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C = \frac{1}{2\sqrt{7}}(\theta + \sin\theta\cos\theta) + C$$

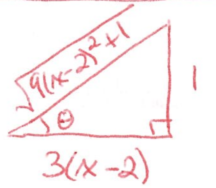
$$= \boxed{\frac{1}{2\sqrt{7}}\left(\sin^{-1}(\sqrt{7}w) + \sqrt{7}w\cdot\sqrt{1-7w^2}\right) + C}$$

4. $\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx$

$9x^2 - 36x + 37 = 9(x^2 - 4x + 4) + 37 - 36$
$= 9(x-2)^2 + 1$

$$\| $$

$$\int \frac{1}{\sqrt{9(x-2)^2+1}}\,dx$$



$\csc\theta = \sqrt{9(x-2)^2+1}$

$\cot\theta = 3(x-2)$

$\cot\theta = 3x - 6$

$\frac{1}{3}\cot(\theta) + 2 = x$

$dx = -\frac{1}{3}\csc^2\theta\, d\theta$

$$= \int \frac{1}{\csc\theta}\cdot\left(-\frac{1}{3}\csc^2\theta\, d\theta\right)$$

$$= -\frac{1}{3}\int \csc\theta\, d\theta$$

$$= -\frac{1}{3}\int \frac{\csc\theta(\csc\theta + \cot\theta)}{(\csc\theta+\cot\theta)}\, d\theta$$

$$= -\frac{1}{3}\int \frac{\csc^2\theta + \cot\theta\csc\theta}{\csc\theta + \cot\theta}\, d\theta$$

let $u = \csc\theta + \cot\theta$
$du = -(\csc\theta\cot\theta + \csc^2\theta)d\theta$
$-du = (\csc\theta\cot\theta + \csc^2\theta)d\theta$

$$= \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|\csc\theta + \cot\theta| + C$$

$$= \boxed{\frac{1}{3}\ln\left|\sqrt{9(x-2)^2+1} + 3(x-2)\right| + C}$$

5. $\int_{-1}^{0} \frac{t^2+7t}{(t+2)(t-1)(t-4)}dt$

split up: $\frac{t^2+7t}{(t+2)(t-1)(t-4)} = \frac{A}{t+2} + \frac{B}{t-1} + \frac{C}{t-4}$

$$t^2+7t = A(t-1)(t-4) + B(t+2)(t-4) + C(t+2)(t-1)$$

$$t^2+7t = A(t^2-5t+4) + B(t^2-2t-8) + C(t^2+t-2)$$

$$t^2+7t = t^2(A+B+C) + t(-5A-2B+C) + (4A-8B-2C)$$

$A+B+C=1$ $\quad$ $-5A-2B+C=7$ $\quad$ $4A-8B-2C=0$

$C=1-A-B$

$-6A-3B+1=7$ $\quad$ $6A-6B-2=0$

$-6A-3B=6$

$-2A-B=2$

$B=-2A-2$

$6A-6(-2A-2)-2=0$

$18A+12-2=0$

$A = \frac{-10}{18} = \frac{-5}{9}$

$B = \frac{20}{18} - \frac{36}{18} = \frac{-16}{18} = \frac{-8}{9}$

$C = \frac{9}{9} + \frac{5}{9} + \frac{8}{9} = \frac{22}{9}$

$$-\frac{5}{9}\int_{-1}^{0}\frac{1}{t+2}dt + \frac{-8}{9}\int_{-1}^{0}\frac{1}{t-1}dt + \frac{22}{9}\int_{-1}^{0}\frac{1}{t-4}dt$$

$$= -\frac{5}{9}\left(\ln|t+2|\Big|_{-1}^{0}\right) - \frac{8}{9}\left(\ln|t-1|\Big|_{-1}^{0}\right) + \frac{22}{9}\left(\ln|t-4|\Big|_{-1}^{0}\right)$$

$$= \boxed{-\frac{5}{9}(\ln(2)-\ln(1)) - \frac{8}{9}(\ln(1)-\ln(2)) + \frac{22}{9}(\ln(4)-\ln(5))}$$

6. $\int \frac{4x-11}{x^3-9x^2}dx$

$\frac{4x-11}{x^2(x-9)}$

$\frac{4x-11}{x^2(x-9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-9}$

$4x-11 = Ax(x-9) + B(x-9) + Cx^2$

$4x-11 = A(x^2-9x) + Bx - 9B + Cx^2$

$4x-11 = x^2(A+C) + x(-9A+B) - 9B$

$A+C=0$ $\quad$ $-9A+B=4$ $\quad$ $-9B=-11$

$B=\frac{11}{9}$

$-9A+\frac{11}{9}=4$

$-9A=\frac{25}{9}$

$A=-\frac{25}{81}$

$C=\frac{25}{81}$

$$\therefore \int\frac{4x-11}{x^3-9x^2}dx = \frac{-25}{81}\int\frac{1}{x}dx + \frac{11}{9}\int\frac{1}{x^2}dx + \frac{25}{81}\int\frac{1}{x-9}dx$$

$$= \boxed{\frac{-25}{81}\ln|x| - \frac{11}{9}\cdot\frac{1}{x} + \frac{25}{81}\ln|x-9| + C}$$

7. $\int \frac{x^5}{x^4+5x^2+4}dx$

$$x^4+5x^2+4 \,\overline{)\, x^5} = x + \frac{-5x^3-4x}{x^4+5x^2+4}$$

$$-(x^5+5x^3+4x)$$

$$-5x^3-4x$$

$$= \int \left(x + \frac{-5x^3-4x}{(x^2+1)(x^2+4)} \right) dx$$

partial fractions:

$$\frac{-5x^3-4x}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$-5x^3-4x = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$-5x^3-4x = Ax^3+Bx^2+4Ax+4B + Cx^3+Dx^2+Cx+D$$

$$-5x^3-4x = x^3(A+C)+x^2(B+D)+x(4A+C)+4B+D$$

$\rightarrow A+C=-5$ $\quad B+D=0$ $\quad 4A+C=-4$ $\quad 4B+D=0$

$A=-C-5$

$B+D=0$ $\quad -3C-20=-4$ $\quad 4B+D=0$

$B=-D$ $\quad -3C=16$ $\quad -3D=0$

$A=\frac{16}{3}-\frac{15}{3}=\frac{1}{3}$ $\quad B=0$ $\quad C=\frac{-16}{3}$ $\quad D=0$

$$= \int x\,dx + \frac{1}{3}\int \frac{x}{x^2+1}dx - \frac{16}{3}\int \frac{x}{x^2+4}dx$$

let $u=x^2+1$, $du=2x\,dx$, $\frac{1}{2}du = x\,dx$

let $u=x^2+4$, $du=2x\,dx$, $\frac{1}{2}du = x\,dx$

$$= \int x\,dx + \frac{1}{3}\cdot\frac{1}{2}\int \frac{1}{u}du + \frac{-16}{3}\cdot\frac{1}{2}\int\frac{1}{u}du$$

$$= \boxed{\frac{x^2}{2} + \frac{1}{6}\ln|x^2+1| - \frac{8}{3}\ln|x^2+4| + C}$$