Math 2300: Calculus	**Spring 2019**

Lecture 9: Friday January 25

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WebAssign due tonight

9.1 Partial Fractions

9.1.1 Last Class...

Last class we learned out to do trig substitutions. In summary,

If you need to get rid of...	Use...
$\sqrt{a^2+x^2}$	$x = a\tan(\theta)$
$\sqrt{a^2-x^2}$	$x = a\sin(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta)$

9.1.2 Warmup

Evaluate the following integrals:

$$\int_0^1 \frac{1}{2x+1}dx = \frac{1}{2}\ln(3)$$

$$\int \frac{1}{(x-7)^2}dx = \frac{(x-7)^{-1}}{-1} + C = \frac{-1}{x-7} + C$$

Yesterday you got an introduction to the method of partial fractions. This can be useful when integrating rational functions that don't have a straightforward anti-derivative. There are three types that we need to be concerned with, and mixtures are allowed: Also, note that in order for these formulas to work, **the**

Linear Factors:	$\frac{5x+3}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$
Repeated Linear Factors:	$\frac{2x}{(x-2)(x+3)^2} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$
Irreducible Quadratic Factors:	$\frac{2x^2-3x-1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

degree of the numerator must be less than the degree of the denominator. You might need to do polynomial long division in order to insure that this happens. Let's do some examples of each type:

9.1.3 Example 1:

$$\int \frac{4}{x^2+5x-14}dx$$

We can't do this one as is, and the numerator degree is less than the denominator degree, so we are ready to use the method of partial fractions. Start by factoring the denominator:

$$\int \frac{4}{(x+7)(x-2)}dx$$

Now, take the integrand and re-write it as the sum of two fractions:

$$\frac{4}{(x+7)(x-2)} = \frac{A}{x+7} + \frac{B}{x-2}$$

Clear denominators by multiplying on both sides by $(x+7)(x-2)$:

$$4 = A(x-2) + B(x+7)$$

Distribute and combine like terms:

$$4 = Ax + Bx - 2A + 7B$$
$$4 = x(A+B) + (7B-2A)$$

Match up the terms on the right and left. On the left, the coefficient of x is 0, so the coefficient of x on the right must be 0:

$$A + B = 0$$

On the left, the constant term is 4, so the constant term on the right must be 4:

$$4 = 7B - 2A$$

Solving this system of equations:

$$\begin{aligned} A + B &= 0 \\ A &= -B \\ 4 &= 7B - 2A \\ 4 &= 7B + 2B \\ 4 &= 9B \\ \frac{4}{9} &= B \\ \frac{-4}{9} &= A \end{aligned}$$

In summary:

$$\frac{4}{(x+7)(x-2)} = \frac{-4/9}{x+7} + \frac{4/9}{x-2}$$

Put this back into the original integral, and solve:

$$\begin{aligned} \int \frac{4}{(x+7)(x-2)}dx &= \int \frac{-4/9}{x+7}dx + \int \frac{4/9}{x-2}dx \\ &= \frac{-4}{9}\int \frac{1}{x+7}dx + \frac{4}{9}\int \frac{1}{x-2}dx \\ &= \frac{-4}{9}\ln|x+7| + \frac{4}{9}\ln|x-2| + C \end{aligned}$$

9.1.4 Example 2

$$\int \frac{x^3}{x^2+2x+1} dx$$

We don't have the degree in the numerator strictly less than the degree in the denominator! Time to do polynomial long division.

$$x^3 \div (x^2+2x+1) = (x-2) + \frac{3x+2}{x^2+2x+1}$$

Now we can split up the integral:

$$\int \frac{x^3}{x^2+2x+1} dx = \int (x-2) dx + \int \frac{3x+2}{x^2+2x+1} dx$$

We can solve the first integral, and then solve the second using partial fractions:

$$\int \frac{x^3}{x^2+2x+1} dx = \frac{x^2}{2} - 2x + C + \int \frac{3x+2}{x^2+2x+1} dx$$

Begin by factoring the integrand we'll use partial fractions on:

$$\frac{3x+2}{x^2+2x+1} = \frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Clear the denominators:

$$3x+2 = A(x+1) + B$$

Distribute and combine like terms:

$$3x+2 = Ax + A + B$$

Since the coefficient of x on the left is 3, the coefficient of x on the right mustbe 3: $A = 3$.
Since the constant term on the left is 2, the constant term on the right must be 2:

$$A + B = 2 \Rightarrow 3 + B = 2 \Rightarrow B = -1.$$

In summary:

$$\frac{3x+2}{(x+1)^2} = \frac{3}{x+1} + \frac{-1}{(x+1)^2}$$

Plugging this onto our integral, now we can integrate:

$$\int \frac{3x+2}{(x+1)^2} dx = \int \frac{3}{x+1} dx + \int \frac{-1}{(x+1)^2} dx = 3\ln|x+1| - \frac{(x+1)^{-1}}{-1} + C$$

And putting this back into our full problem, we get:

$$\int \frac{x^3}{x^2+2x+1} dx = \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C$$

9.1.5 Example 3:

$$\int \frac{10}{(x+1)(x^2+9)}dx$$

This one is immediately partial fractions. Use the setup with an irreducible quadratic:

$$\frac{10}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Clear denominators:

$$10 = A(x^2+9) + (Bx+C)(x+1)$$

Distribute and combine like terms:

$$10 = Ax^2 + 9A + Bx^2 + Bx + Cx + C = (A+B)x^2 + (B+C)x + (9A+C)$$

Since the coefficients of x^2 and x on the left are 0, so must be the coefficients on the right:

$$A+B=0,\ B+C=0.$$

Since the constant term on the left is 10, so must be the constant term on the right:

$$9A+C=0$$

Solving this system of equations:

$$A=-B,\ C=-B \Rightarrow C=A,\ 9A+C=10 \Rightarrow A=C=1,\ B=-1.$$

In summary:

$$\frac{10}{(x+1)(x^2+9)} = \frac{1}{x+1} + \frac{-x+1}{x^2+9}$$

Now we can split up and integrate:

$$\int \frac{10}{(x+1)(x^2+9)}dx = \int \frac{1}{x+1}dx + \int \frac{-x+1}{x^2+9}dx = \int \frac{1}{x+1}dx + \int \frac{-x}{x^2+9}dx + \int \frac{1}{x^2+9}dx$$

The first integral is straightforward. The middle one requires u-substitution. The third integral is an arctan integral.
Integral 1: $\int \frac{1}{x+1}dx = \ln|x+1| + C$
Integral 2: Let $u = x^2+9$, then $du = 2xdx$, so $\frac{1}{2}du = xdx$. Subbing in:

$$\int \frac{-x}{x^2+9}dx = \frac{-1}{2}\int \frac{1}{u}du = \frac{-1}{2}\ln|u| + C = \frac{-1}{2}\ln|x^2+9| + C$$

Integral 3:

$$\int \frac{1}{x^2+9}dx = \int \frac{1}{9}\cdot\frac{1}{\frac{x^2}{9}+1}dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}dx = \frac{1}{9}\cdot 3\arctan\left(\frac{x}{3}\right) + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Putting these all together:

$$\int \frac{10}{(x+1)(x^2+9)}dx = \ln|x+1| + \frac{-1}{2}\ln|x^2+9| + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$