Lecture 7:

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7.1 Trigonometric Substitutions

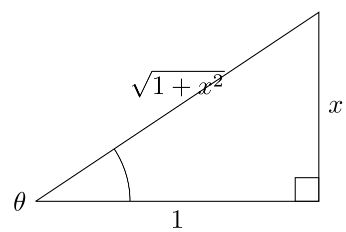
In this section, we consider cases where it is useful to substitute in a trig expression for the variable in an integral. We will use triangles and the Pythagorean theorem to do this. *These are hard. There are not enough practice problems in your WebAssign and Homework. Seek other practice - I will link to some on my website.*

7.1.1 How to substitute in $\sqrt{a^2+x^2}$

Integrate:

$$\int \frac{1}{\sqrt{1+x^2}}dx.$$

Draw a triangle with $\sqrt{a^2+x^2}$ as the hypoteneuse. In the case we are working on, $a=1$:



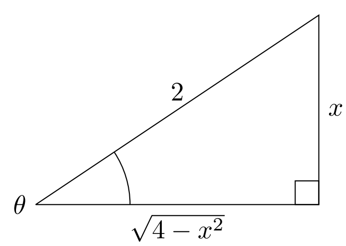
Now, we see that $\cos(\theta)=\frac{1}{\sqrt{1+x^2}}$, and $\tan(\theta)=x$.
Notices that this means $dx=\sec^2(\theta)d\theta$. Using these substitutions in the integral:

$$\begin{aligned}
\int \frac{1}{\sqrt{1+x^2}}dx &= \int \cos(\theta)\sec^2(\theta)d\theta \\
&= \int \sec(\theta)d\theta \\
&= \int \frac{\sec(\theta)(\sec(\theta)+\tan(\theta))}{(\sec(\theta)+\tan(\theta))}d\theta \\
&= \int \frac{\sec^2(\theta)+\sec(\theta)\tan(\theta)}{(\sec(\theta)+\tan(\theta))}d\theta \\
&\text{Let } u=\sec(\theta)+\tan(\theta);\ du=(\sec(\theta)\tan(\theta)+\sec^2(\theta))d\theta \\
&= \int \frac{1}{u}du \\
&= \ln|\sec(\theta)+\tan(\theta)|+C \\
&= \ln\left|\sqrt{1+x^2}+x\right|+C
\end{aligned}$$

7.1.2 How to substitute in $\sqrt{a^2 - x^2}$

$$\int \sqrt{4 - x^2} dx.$$

Draw the triangle with $a = 2$ as the hypoteneuse, as we always ened 4 to be bigger than x^2 for the square root to be valid:



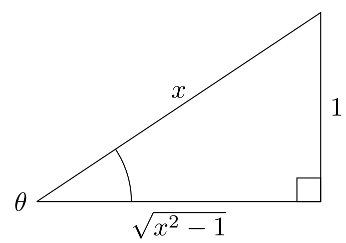
Notice $2\cos(\theta) = \sqrt{4 - x^2}$.
We also have $2\sin(\theta) = x$, so $dx = 2\cos(\theta)d\theta$. Plugging this into the integral:

$$\begin{aligned}
\int \sqrt{4 - x^2} dx &= \int 2\cos(\theta) 2\cos(\theta) d\theta \\
&= 4 \int \cos^2(\theta) d\theta \\
&\text{Since } \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)) : \\
&= 2 \int (1 + \cos(2\theta)) d\theta \\
&= 2(\theta + \frac{1}{2}\sin(2\theta)) + C \\
&= 2\theta + \sin(2\theta) + C \\
&= 2\theta + 2\sin(\theta)\cos(\theta) + C \\
&= 2\sin^{-1}(x) + 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4 - x^2}}{2} + C \\
&= 2\sin^{-1}(x) + \frac{x\sqrt{4 - x^2}}{2} + C
\end{aligned}$$

7.1.3 How to substitute in $\sqrt{x^2 - a^2}$

$$\int \frac{1}{(x^2-1)^{3/2}} dx$$

First, draw a triangle. This time x is the hypotenuse:

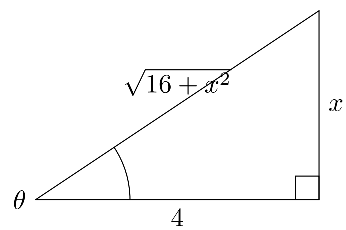


Notice, $\tan(\theta) = \frac{1}{\sqrt{x^2-1}}$ and $\csc(\theta) = x$.
This means $dx = -\csc(\theta)\cot(\theta)d\theta$.
Substituting in:

$$\begin{aligned}
\int \frac{1}{(x^2-1)^{3/2}} dx &= \int \tan^3(\theta)(-\csc(\theta)\cot(\theta))d\theta \\
&= -\int \frac{\sin^2(\theta)}{\cos^2(\theta)} \frac{1}{\sin(\theta)} d\theta \\
&= -\int \frac{\sin(\theta)}{\cos^2(\theta)} d\theta \\
&\text{Let } u = \cos(\theta), du = -\sin(\theta)d\theta : \\
&= \int u^{-2} du \\
&= -u^{-1} + C \\
&= \frac{-1}{\cos(\theta)} + C \\
&= -\sec(\theta) + C \\
&= \frac{-x}{\sqrt{x^2-1}} + C
\end{aligned}$$

7.1.3.1 Extra Example:

Evaluate $\int \frac{\sqrt{x^2+16}}{x^4} dx$. Begin using a triangle:



Now plug into the integral:

$$\begin{aligned}
\int \frac{\sqrt{x^2+16}}{x^4} dx &= \int \frac{4\sec(\theta)}{4^4\tan^4(\theta)} \cdot 4\sec^2(\theta) d\theta \\
&= \frac{1}{16} \int \frac{\sec^3(\theta)}{\tan^4(\theta)} d\theta \\
&= \frac{1}{16} \int \frac{1}{\cos^3(\theta)} \frac{\cos^4(\theta)}{\sin^4(\theta)} d\theta \\
&= \frac{1}{16} \int \frac{\cos(\theta)}{\sin^4(\theta)} d\theta \\
&\text{Let } u = \sin(\theta), \text{ then } du = \cos(\theta): \\
&= \frac{1}{16} \int u^{-4} du \\
&= \frac{1}{16} \frac{u^{-3}}{-3} + C \\
&= \frac{-1}{48} \frac{1}{\sin^3(\theta)} + C \\
&= \frac{-1}{48} \frac{1}{\left(\frac{x}{\sqrt{16+x^2}}\right)^3} + C \\
&= \frac{-(\sqrt{16+x^2})^3}{48x^3} + C
\end{aligned}$$