Math 2300: Calculus **Spring 2019**

Lecture 6: Tuesday January 22

Lecturer: Sarah Arpin *Scribes:*

WebAssign due tonight

6.1 Warm-up

$$\int 2^x \cos(x) dx$$

6.2 Trigonometric Integrals (5.7)

Today we're going to learn three techniques to evaluate integrals involving trig functions.

1. How to use trig identities to help evaluate trig integrals
2. How to choose a good u to use u-substitution in evaluating trig integrals.
3. How to make good u, dv choices to use integration by parts in evaluating trig integrals.

6.2.1 Trig Identities

You should be able to use these easily:

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\tan^2(x) + 1 = \sec^2(x)$
3. $1 + \cot^2(x) = \csc^2(x)$
4. $\sin^2(x) = \frac{1-\cos(2x)}{2}$
5. $\cos^2(x) = \frac{1+\cos(2x)}{2}$

6.2.1.1 Example:

$$\begin{aligned}
\int \sin^2(\theta) d\theta &= \int \frac{1-\cos(2\theta)}{2} d\theta \\
&= \int \frac{1}{2} d\theta - \frac{1}{2} \int \cos(2\theta) d\theta \\
&= \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C
\end{aligned}$$

6.2.2 Choosing a good u

6.2.2.1 Products of Powers of Sine and Cosine

- If at least one power is odd, pull one of these out to put with the dx and convert the remaining even powers to an expression involving the other trig function using pythagorean identities. Let $u =$ the antiderivative of this trig function.

  For example:

$$\begin{aligned}\int \sin^4(x)\cos^5(x)dx &= \int \sin^4(x)\cos^4(x)\cos(x)dx \\ &= \int \sin^4(x)(\cos^2(x))^2\cos(x)dx \\ &= \int \sin^4(x)(1-\sin^2(x))^2\cos(x)dx \\ &\text{Let } u=\sin(x)\text{, so } du=\cos(x)dx \\ &= \int u^4(1-u^2)^2du \\ &= \int u^4(1-2u^2+u^4)du \\ &= \int (u^4-2u^6+u^8)du \\ &= \frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}+c \\ &= \frac{(\sin(x))^5}{5}-\frac{2(\sin(x))^7}{7}+\frac{(\sin(x))^9}{9}+c\end{aligned}$$

  **Note, we still could have used this if it was $\int \sin^5(x)\cos^5(x)dx$! We use this method anytime *at least* one of the powers is odd.

- If both powers are even, use the power-reducing identity, maybe twice.

  For example:

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int \frac{1-\cos(2x)}{2}\frac{1+\cos(2x)}{2}dx \\ &= \frac{1}{4}\int (1-\cos^2(2x))dx \\ &= \frac{1}{4}\int 1dx - \frac{1}{4}\int \cos^2(2x)dx \\ &= \frac{1}{4}\int 1dx - \frac{1}{4}\int \frac{1+\cos(4x)}{2}dx \\ &= \frac{1}{8}\int 1dx - \frac{1}{8}\int \cos(4x)dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x)+C\end{aligned}$$

6.2.2.2 Products of Powers of Secant and Tangent

- If power on $\sec(x)$ is even: Use $u = \tan(x)$, and put aside $\sec^2(x)dx$ part of integrand. Convert remaining even powers of secant to tangent using the pythagorean identity.

For example:

$$\begin{aligned}
\int \tan^4(x)\sec^2(x)dx &= \int \tan^4(x)\sec^2(x)dx \\
&\text{Let } u = \tan(x) \text{, so } du = \sec^2(x)dx\text{:} \\
&= \int u^4 du \\
&= \frac{u^5}{5} + C \\
&= \frac{\tan^5(x)}{5} + C
\end{aligned}$$

- If power on $\tan(x)$ is odd and $\sec(x)$ is odd: Use $u = \sec(x)$, put aside $\tan(x)\sec(x)dx$ part of the integral and convert the remaining even powers of tangent to secant using the pythagorean identity.

  For example:

$$\begin{aligned}
\int \tan^3(x)\sec^5(x)dx &= \int \tan^2(x)\sec^4(x)\tan(x)\sec(x)dx \\
&= \int (\sec^2(x) - 1)\sec^4(x)\tan(x)\sec(x)dx \\
&\text{Let } u = \sec(x) \text{, so } du = \tan(x)\sec(x)dx\text{:} \\
&= \int (u^2 - 1)u^4 du \\
&= \int (u^6 - u^4)du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + C
\end{aligned}$$

- If power on $\sec(x)$ is odd and power on $\tan(x)$ is even...bleh. It's going to be long and tricky and likely require integration by parts and u-substitution. This is not ideal...

6.2.3 Choosing a good u, dv

This one is difficult, so we have a difficult example to work with. Try to use integration by parts on:

$$\begin{aligned}
\int \sec^3(x)dx &= \int \sec(x)\sec^2(x)dx \\
&\text{Let } u = \sec(x)\text{, } dv = \sec^2(x)dx\text{;} \\
&\text{then } du = \sec(x)\tan(x)dx \text{ and } v = \tan(x)\text{:} \\
&= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx \\
&= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx \\
2\int \sec^3(x)dx &= \sec(x)\tan(x) + \int \sec(x)dx
\end{aligned}$$

And we learned that $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$ from a previous worksheet, so we have:

$$2\int \sec^3(x)dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$$
$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$$

In summary:

Situation	How to start?
$\int \sin^{\text{odd}}(x)\cos^n(x)dx$	Do u-sub with $u = \cos(x)$
$\int \sin^n(x)\cos^{\text{odd}}(x)dx$	Do u-sub with $u = \sin(x)$
$\int \sin^{\text{even}}(x)\cos^{\text{even}}(x)dx$	Use power-reducing formulas
$\int \tan^n(x)\sec^{\text{even}}(x)dx$	Do u-sub with $u = \tan(x)$
$\int \tan^{\text{odd}}(x)\sec^{\text{odd}}(x)dx$	Do u-sub with $u = \sec(x)$
$\int \tan^{\text{even}}(x)\sec^{\text{odd}}(x)dx$	Going to be complicated...boomerang.

Note: 0 is an even number.