Math 2300: Calculus **Spring 2019**

Lecture 5: Friday January 18

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Reminder: There is a WebAssign homework due tonight at 11:59pm, and a written homework due tomorrow in class with your TA.

5.1 Day 5: Integration by Parts II

5.1.1 Finish Examples From Last Lecture:

1. $\int x\cos(x)dx$
   Hint: $u = x$, $dv = \cos(x)dx$
   Solution: $x\sin(x) + \cos(x) + C$
2. $\int 4xe^{-2x}dx$
   Hint: $u = 4x$, $dv = e^{-2x}dx$
   Solution: $-2xe^{-2x} - e^{-2x} + C$
3. $\int_6^0 (2+5x)e^{\frac{1}{3}x}dx$
   Solution: $(15xe^{\frac{1}{3}x} - 39e^{\frac{1}{3}x})|_6^0 = -39 - 51e^2$
4. $\int \sqrt{x}\ln(x)dx$
   Hint: $u = \ln(x)$, $dv = \sqrt{x}dx$
   Solution: $\frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2} + C$
5. $\int x \cdot 2^x dx$
   Hint: $u = x$, $dv = 2^x dx$
   Solution: $\frac{x\cdot 2^x}{\ln(2)} - \frac{2^x}{(\ln(2))^2} + C$

5.1.2 Trick 1: Sometimes, you have to do more than one round...

For example, consider:

$$\int x^2\cos(x)dx$$

We can't integrate directly, so let's try integration by parts:

$$u = x^2,\ dv = \cos(x)dx$$

$$du = 2x,\ v = \sin(x)$$

Then, plug into the formula and simplify:

$$\int udv = uv - \int vdu$$

$$\begin{aligned}\int x^2\cos(x)dx &= x^2\sin(x) - \int 2x\sin(x)dx \\ &= x^2\sin(x) - 2\int x\sin(x)dx\end{aligned} \tag{5.1}$$

$x\sin(x)$ is certainly simpler than $x^2\cos(x)$, but we still can't integrate it. However, if we use integration by parts on $\int x\sin(x)$, we do get an expression that we can integrate. So, let's leave (5.1) for a moment and just focus on $\int x\sin(x)dx$:

We get a new u and dv here:

$$u = x,\ dv = \sin(x)$$

$$du = dx,\ v = -\cos(x)dx$$

And plug into the integration by parts formula:

$$\begin{aligned}\int x\sin(x)dx &= -x\cos(x) - \int(-\cos(x))dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned} \tag{5.2}$$

Now, we can take this result and plug it into (5.1) to finish the problem:

$$\begin{aligned}\int x^2\cos(x)dx &= x^2\sin(x) - 2\int x\sin(x)dx \\ &= x^2\sin(x) - 2(-x\cos(x) + \sin(x) + C) \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C\end{aligned}$$

5.1.3 Trick 2: Sometimes, you have to re-arrange things to solve for the right value:

This sometimes happens with integrals involving sin and cos factors, since their derivatives just cycle around. Consider $\int \sin(x)e^x dx$.

$$u = \sin(x),\ dv = e^x dx$$

$$du = \cos(x)dx,\ v = e^x$$

Plugging this into our formula:

$$\int \sin(x)e^x dx = \sin(x)e^x - \int \cos(x)e^x dx$$

We'll need one more round, since the integral isn't something we can just evaluate:

$$\int \cos(x)e^x dx$$

$$u = \cos(x),\ dv = e^x dx$$

$$du = -\sin(x)dx,\ v = e^x$$

$$\int \cos(x)e^x dx = \cos(x)e^x + \int \sin(x)e^x dx$$

This might not look promising, but notice that the integral is the **same as our original.** Plugging this in above:

$$\int \sin(x)e^x dx = \sin(x)e^x - \int \cos(x)e^x dx$$
$$= \sin(x)e^x - (\cos(x)e^x + \int \sin(x)e^x dx)$$
$$= \sin(x)e^x - \cos(x)e^x - \int \sin(x)e^x dx$$
$$\text{Add} \int \sin(x)e^x dx \text{ to both sides:}$$
$$2\int \sin(x)e^x dx = \sin(x)e^x - \cos(x)e^x + C$$
$$\int \sin(x)e^x dx = \frac{1}{2}\sin(x)e^x - \frac{1}{2}\cos(x)e^x + C$$

We did it!

When to use this: when you're using integration by parts, and you notice that the integral you end up with might cycle around to be your original integral. The u will likely be a trig function (like sin or cos), and the dv will be something where v is the same as (or close to) dv.