Math 2300: Calculus **Spring 2019**

Lecture 3: Wednesday January 16

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Day 3: Integration by Parts (Part I)

3.0.1 A word of warning...

It's important to be comfortable with u-substitution by now. You want to be able to do some of the simpler ones in your head. For example:

$$\int \cos(2x)dx = \frac{1}{2}\sin(2x) + C,$$

$$\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + C.$$

If this is not the case, make sure to do some more practice on this topic tonight. You can google "u-substitution integral worksheets with solutions" or something similar and see what you find. You can also email me and I can do that too.

3.0.2 The Big Picture

Integration by parts is just another tool to help us evaluate difficult integrals.

This tool is based on the derivative product rule from Calc 1. Remember:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

All we do from here is integrate to get the integration by parts formula:

$$\int (f(x)g(x))'dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\Rightarrow f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

In this form, the rule might not seem so useful. But, with some re-arranging we get a formula that solves for an integral:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

So we are looking for situations where the integrand is made up of two factors, $f(x)$ and $g'(x)$, where $f'(x)g(x)$ is significantly easier to integrate.

Another way of writing this formula uses u's and v's:

$$\int udv = uv - \int vdu.$$

In this formula, $f(x)$ corresponds to u, so $f'(x)$ corresponds to du, and $g(x)$ corresponds to v, so $g'(x)$ corresponds to dv. **Make sure this makes sense.**

3.0.3 An Example

Let's see how the mechanics actually work with an example. Consider:

$$\int xe^{6x}dx.$$

Try for a minute to integrate it using more familiar techniques...it's not going to work out! We can't do a u-sub and we don't know how to generically integrate products.

But, we do have the integration by parts formula. Here's how it lines up:

$$\int xe^{6x}dx \Rightarrow u = x,\ dv = e^{6x}dx$$

That's one way of splitting up the integrand into factors. Since the derivative of x is easier than x, we've made the right choice. Notice:

$$du = 1dx,\ v = \frac{1}{6}e^{6x}$$

Now let's use the formula:

$$\int udv = uv - \int vdu$$

$$\begin{aligned}
\int xe^{6x}dx &= x \cdot \frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x}dx \\
&= x \cdot \frac{1}{6}e^{6x} - \frac{1}{6}\int e^{6x}dx \\
&= x \cdot \frac{1}{6}e^{6x} - \frac{1}{6}\left(\frac{1}{6}e^{6x} + C\right) \\
&= x \cdot \frac{1}{6}e^{6x} - \frac{1}{36}e^{6x} + C
\end{aligned}$$

3.0.4 Some Tips

In the last example, the key was making the right choice for which part of the integral should be u and which should be dv. In general, u should be a factor whose derivative is simpler than u itself. And dv should be something where it's not too hard to integrate in your head and figure out what v is.

When you're starting out, the choice is not always obvious. That's fine, just go back and make a different choice. Let's see what it looks like when we make the **wrong** choice of u and dv:

$$\int xe^{6x}dx$$

$$u = e^{6x},\ dv = xdx$$

$$du = 6e^{6x},\ v = \frac{1}{2}x^2$$

Plugging this in:

$$\int xe^{6x}dx = e^{6x}\frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot 6e^{6x}dx.$$

That integral on the righthand side is not something we can integrate! Moreover, it's more complicated to integrate than the one we started with!

So when you choose the wrong u and dv, the righthand side of the integration by parts formula is more complicated than what you started with. That's how you know you went wrong, and to go back and choose a different u and dv.

3.0.5 Another Example:

$$\int (3t+5)\cos\left(\frac{t}{4}\right)dt$$

Can't integrate this one directly. What's a good choice for u and dv this time?

$$u = 3t+5,\ dv = \cos\left(\frac{t}{4}\right)dt$$

From these defintions, we get:

$$du = 3dt,\ v = 4\sin\left(\frac{t}{4}\right).$$

Plugging this into our formula and simplifying:

$$\int udv = uv - \int vdu$$

$$\begin{aligned}\int (3t+5)\cos\left(\frac{t}{4}\right)dt &= (3t+5)4\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)\cdot 3dt \\ &= (3t+5)4\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)dt \\ &= (3t+5)4\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + C\end{aligned}$$

3.0.6 What about definite integrals?

We can use a revised integration by parts formula for the definite integral situation:

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

Remember that $uv|_a^b = u(b)v(b) - u(a)v(a)$.

3.0.7 Start with a familiar example:

$$\int_{-1}^{2} xe^{6x}dx$$

Choose the same u and dv as we did last class:

$$u = x,\ dv = e^{6x}dx$$

$$du = 1dx,\ v = \frac{1}{6}e^{6x}$$

Plug this into the formula and evaluate:

$$\begin{aligned}\int_{-1}^{2} xe^{6x}dx &= \frac{1}{6}xe^{6x}|_{-1}^{2} - \int_{-1}^{2}\frac{1}{6}e^{6x}dx \\ &= \frac{1}{6}xe^{6x}|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2} e^{6x}dx \\ &= \frac{1}{6}xe^{6x}|_{-1}^{2} - \frac{1}{6}(\frac{1}{6}e^{6x}|_{-1}^{2}) \\ &= (\frac{1}{6}(2)e^{12} - \frac{-1}{6}e^{-6}) - \frac{1}{6}(\frac{1}{6}e^{12} - \frac{1}{6}e^{-6}) \\ &= \frac{1}{3}e^{12} + \frac{1}{6}e^{-6} - \frac{1}{36}e^{12} + \frac{1}{36}e^{-6} \\ &= \frac{11}{36}e^{12} + \frac{7}{36}e^{-6}\end{aligned}$$

3.0.8 Try some on your own

1. $\int x\cos(x)dx$
   Hint: $u = x$, $dv = \cos(x)dx$
   Solution: $x\sin(x) + \cos(x) + C$

2. $\int 4xe^{-2x}dx$
   Hint: $u = 4x$, $dv = e^{-2x}dx$
   Solution: $-2xe^{-2x} - e^{-2x} + C$

3. $\int_6^0 (2 + 5x)e^{\frac{1}{3}x}dx$
   Solution: $(15xe^{\frac{1}{3}x} - 39e^{\frac{1}{3}x})|_6^0 = -39 - 51e^2$

4. $\int \sqrt{x}\ln(x)dx$
   Hint: $u = \ln(x)$, $dv = \sqrt{x}dx$
   Solution: $\frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2} + C$

5. $\int x \cdot 2^x dx$
   Hint: $u = x$, $dv = 2^x dx$
   Solution: $\frac{x\cdot 2^x}{\ln(2)} - \frac{2^x}{(\ln(2))^2} + C$