Math 1300: Calculus I **Fall 2020**

Lecture: Section 5.5: The Substitution Rule

Lecturer: Sarah Arpin

Today's Goal: Learn an integration technique called "u-substitution"

Logistics: We start this on a Tuesday (12/1), after an activity, and finish on Wednesday.

Warm-Up 1.1 (True or False:) *If $f(x)$ is continuous on $[a, b]$, then:*

$$\int_a^b x f(x) dx = x \int_a^b f(x) dx.$$

1.1 Chain Rule

With all of our antiderivative practice, we still have difficulty un-doing chain rule, especially more complicated versions. For Example:

$$\frac{d}{dx} \sin(x^2 - 2x) =$$

Which means we can actually integrate:

$$\int (2x - 2) \cos(x^2 - 2x) dx =$$

But if we had been given $\int (2x - 2) \cos(x^2 - 2x) dx$ without the previous exercise, it certainly would have been hard to recognize. We develop a technique that will make this easier, called **u-substitution**.

1.2 Indefinite Integrals With Substitution

Theorem 1.2 (The Substitution Rule) *If $u = g(x)$ is differentiable, then:*

$$\int g'(x)f(g(x))dx = \int f(u)du$$

The procedure is best summed up in a few steps, alongside an example:

Example 1.3

$$\int e^{\tan(x)} \sec^2(x)dx$$

1. *In your integral, identify a factor that looks like a composition of two functions. Which is the inner function? Which is the outer function?*

2. *Do you see the derivative of the inner function elsewhere in the function? We are looking for an expression of the form $f(g(x)) \cdot g'(x)$.*

3. *Let $u = g(x)$.*

4. *Find $\frac{du}{dx}$:*

5. *Find an expression for dx in terms of du:*

6. *Replace everything in the integral that has to do with x with expressions involving u. In particular, $u = g(x)$ and $du = g'(x)dx$.*

7. *Evaluate the integral with respect to u*

8. *Un-do the replacement, and replace u with $g(x)$.*

Example 1.4

$$\int (x^4 - 1)^{10} x^3 dx$$

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2. *Do you see the derivative of the inner function elsewhere in the function? We are looking for an expression of the form $f(g(x)) \cdot g'(x)$.*

3. *Let $u = g(x)$.*

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6. *Replace everything in the integral that has to do with x with expressions involving u. In particular, $u = g(x)$ and $du = g'(x)dx$.*

7. *Evaluate the integral with respect to u.*

8. *Un-do the replacement, and replace u with $g(x)$.*

Example 1.5

$$\int x \cos(x^2) dx$$

1.3 Definite Integrals With Substitution

With definite integrals, we can use the same process but we need to be careful about the bounds: remember that they define x-values, not u-values, so you want to return to x before using the evaluation theorem:

Example 1.6

$$\int_1^2 \frac{\ln(x)}{x} dx$$

Example 1.7

$$\int_{-1}^1 \frac{e^x}{e^x - 5} dx$$

1.4 Symmetry

Suppose f is continuous on $[-a, a]$.

(a) If f is even (so $f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)dx =$$

(b) If f is odd (so $f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)dx =$$

1.4.1 Common Even Functions

(Don't forget trig functions!)

1.4.2 Common Odd Functions

1.4.3 Examples

Example 1.8

$$\int_{-3}^{3} \frac{\sin(x)}{x^4 + 2x^2 + 3} dx$$