Math 1300: Calculus I **Fall 2020**

Lecture: Section 5.3: Evaluating Definite Integrals

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Today's Goal: A new technique to evaluate definite integrals.

Logistics: We will start this on Tuesday and finish it on Wednesday. There is a check-in on Friday (always!), and we will start section 5.4 on Friday as well.

Warm-Up 1.1 *Using geometry, evaluate the integral* $\int_4^0 \sqrt{16 - x^2} dx$.

(A) 2π

(B) 4π

(C) 8π

(D) 16π

(E) None of the above

So far, we have two methods for evaluating deifinite integrals:

1. Geometry
2. Taking a limit of a Riemann sum

Today, we will learn another technique, and this technique ties areas under curves back to antiderivatives that we had practiced in Section 5.1.

This method is part of the Fundamental Theorem of Calculus, which we will talk about more in 5.4. It is sometimes known as the "Evaluation Theorem".

Theorem 1.2 (Fundamental Theorem of Calculus Part II) *Suppose f is continuous on $[a, b]$ and F is any antiderivative of f (i.e., $F' = f$). Then,*

$$\int_b^a f(x)dx = F(b) - F(a).$$

Example 1.3 *Use the evaluation theorem to evaluate the integral:*

$$\int_2^3 2^x dx$$

1.0.1 Jusitfying the Evaluation Theorem

Theorem 1.4 (Fundamental Theorem of Calculus Part II) *Suppose f is continuous on $[a, b]$ and F is any antiderivative of f (i.e., $F' = f$). Then,*

$$\int_b^a f(x)dx = F(b) - F(a).$$

Recall that the definite integral is defined as a limit of a Riemann sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Lets consider one of these infinitely small intervals in particular. We will have to use a stretch of imagination, as *infinitely small* width is not easy to imagine.

But let's suppose $[x_i, x_{i+1}]$ is an infinitely small interval, and that $f(x_i^*)$ is the height of our infinitely small rectangle.

Recall that $f = F'$, and consider the Intermediate Value Theorem in this context:

$$\frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} = F'(c) = f(c)$$

for some c in $[x_i, x_{i+1}]$.

What if our c was x_i^*? Then the IVT tells us:

$$\frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i} = f(x_i^*)$$

The width of our interval is Δx:

$$\frac{F(x_{i+1}) - F(x_i)}{\Delta x} = f(x_i^*)$$

And if we move the Δx to the other side we have:

Putting these all back in our Riemann sum gives:

Example 1.5 *Use the Evaluation Theorem to evaluate the following integrals:*

(a) $\int_0^{\pi/4} \sin(x)dx$

(b) $\int_{-1}^{1} e^x dx$

(c) $\int_0^{\pi/3} \sec(t)\tan(t)dt$

(d) $\int_{-1}^{0} \frac{1}{1+y^2} dy$

1.1 Indefinite Integrals

In the past, we have simply asked for the family of antiderivatives of a particular function: No notation.

Example 1.6 *Find all possible antiderivatives of the function* $G(x) = x^3 - e^x$.

Now that we have the evaluation theorem and the notation of integrals, we can rephrase this question using a new notation for antiderivatives: the **indefinite integral**:

Example 1.7 *Find* $\int (x^3 - e^x)dx$

Example 1.8 *This is an important example, and has a bit of a twist! Try to remember a discussion we had about this many weeks ago...*
Evaluate:

$$\int \frac{1}{x}dx$$

Example 1.9

$$\int \frac{2\sqrt{x} - 3x^6 + 2\sqrt[3]{x^2} - \pi}{x^2}$$

1.2 Applications

Theorem 1.10 (Net Change Theorem) *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)dx = F(b) - F(a).$$

For example, we know that velocity is the rate of change of position. This tells us:

$$\int_a^b v(t)dt = s(b) - s(a),$$

where s is position and v is velocity, so $s' = v$.

Example 1.11 *A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)dt$ represent?*

1.2.1 Displacement vs. Distance

If we want to calculate the displacement of a moving object over time, we integrate velocity:

$$\int_a^b v(t)dt = s(b) - s(a),$$

so $s(b) - s(a)$ gives the net change in position.

But what if we want to know the **total distance** traveled by the object? In other words, what if we want to count all motion as positive distance?

$$\text{total distance traveled} = \int_a^b |v(t)|dt$$

Example 1.12 *The velocity function is given by* $v(t) = t^2 - 2t - 0$ *for a particle moving along a line. Find both the displacement and the distance traveled by the particle during the time interval* $1 \leq t \leq 6$.

Example 1.13 *Water flows from the bottom of a storage tank at a rate of* $r(t) = 200 - 4t$ *liters per minute, where* $0 \leq t \leq 50$*. Find the amount of water that flows from the tank during the first 10 minutes.*