The first and second Stiefel-Whitney classes; orientation and spin structure

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Classifying spaces and universal bundles

Theorem 1. *Given a topological group G, there is a space BG and a principal G-bundle EG such that for any space B (homotopy equivalent to a CW-complex), the principal G-bundles on B are in bijection with homotopy classes of maps $f : B \to BG$ via pullback. I.e., given a principal G-bundle $E \to B$, there exists a map $f : B \to BG$ such that $f^*EG \cong E$ and $f^*EG \cong g^*EG$ if and only if f and g are homotopy equivalent.*

For $G = O(n)$, we can take $BG = Gr_n(\mathbb{R}^\infty)$, a limit of the Grassmann manifolds $Gr_n(\mathbb{R}^k)$ of n-planes in $\mathbb{R}^k$. Above this we have $EG = V_n(\mathbb{R}^\infty)$, a limit of the Stiefel manifolds $V_n(\mathbb{R}^k)$ of orthogonal n-frames in $\mathbb{R}^k$. The map $EG \to BG$ is given by forgetting the framing. The fiber is clearly $O(n)$.

Stiefel-Whitney classes

Vaguely, characteristic classes are cohomology classes associated to vector bundles (functorially) over a space B. We will be concerned with the **Stiefel-Whitney** classes in $H^*(B;\mathbb{F}_2)$ associated to real vector bundles over B. These are mod 2 reductions of obstructions to finding $(n-i+1)$ linearly independent sections of an n-dimensional vector bundle over the i skeleton of B.

Theorem 2 ([M], Chapter 23)**.** *There are characteristic classes $w_i(\xi) \in H^i(B;\mathbb{F}_2)$ associated to an n-dimensional real vector bundle $\xi : E \to B$ that satisfy and are uniquely determined by*

- $w_0(\xi) = 1$ *and* $w_i(\xi) = 0$ *for* $i > \dim \xi$,
- $w_i(\xi \oplus \epsilon) = w_i(\xi)$ *where* ϵ *is the trivial line bundle* $\mathbb{R} \times B$,
- $w_1(\gamma_1) \neq 0$ *where* γ_1 *is the universal line bundle over* $\mathbb{RP}^\infty$,
- $w_i(\xi \oplus \zeta) = \sum_{j=0}^{i} w_j(\xi) \cup w_{i-j}(\zeta)$.

Every $\mathbb{F}_2$ characteristic class for n-dimensional real vector bundles can be written as a polynomial in $\mathbb{F}_2[w_1, \dots, w_n]$.

One can view these as coming from "universal" Stiefel-Whitney classes in $H^*(BO(n);\mathbb{F}_2)$

Theorem 3 ([M], Chapter 23)**.** *There are classes $w_i \in H^i(BO(n);\mathbb{F}_2)$ that satisfy and are uniquely characterized by the following*

- $w_0 = 1$ *and* $w_i = 0$ *for* $i > n$,
- $w_1 \neq 0$ *for* $n = 1$,
- $i^*(w_i) = w_i$ *where* $i_n : BO(n) \to BO(n+1)$ *is the classifying map for* $EO(n) \to BO(n)$,
- $p^*_{m,n}(w_i) = \sum_{j=0}^{i} w_j \otimes w_{i-j}$ *where* $p_{m,n} : BO(m) \times BO(n) \to BO(m+n)$ *is the classifying map for the product* $EBO(m) \times EBO(n) \to BO(m) \times BO(n)$.

The cohomology ring $H^*(BO(n); \mathbb{F}_2)$ *is the polynomial ring* $\mathbb{F}_2[w_1, \dots, w_n]$. *If* $f : B \to BG$ *is a classifying map for the bundle* $\xi : E \to B$, *then* $w_i(\xi) = f^*(w_i)$.

The exact sequences associated to fibration

A **fibration** $\xi : E \to B$ is a surjective map such that satisfies the homotopy lifting property: given a space Y, $f : Y \to E$, and $h : Y \times I \to B$, there exists $\tilde{h} : Y \times I \to E$ making the following diagram commute

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ {\scriptstyle i_0}\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle \xi} \\ Y \times I & \xrightarrow[h]{} & B \end{array}$$

If $E \to B$ satisfies the homotopy lifting property for all finite dimensional disks, it is called a *Serre fibration*. A fiber bundle $F \to E \xrightarrow{\xi} B$ is a space E that is locally a product: there exists $B = \cup_i U_i$ such that $\xi^{-1}(U_i) \cong U_i \times F$.

Proposition 1. *Fiber bundles are Serre fibrations.*

Given a Serre fibration $\xi : E \to B$ with $F = \xi^{-1}(b_0)$, we obtain a long exact sequence in homotopy (coming for the long exact sequence for the pair (E, F)). So in particular we get a long exact sequence for the homotopy groups of the spaces in a fiber bundle $F \to E \to B$.

Proposition 2 ([H] Theorem 4.41)**.** *If* $F \to E \xrightarrow{\xi} B$ *is a fiber bundle with* $b_0 \in B$ *and* $x_0 \in \xi^{-1}(b_0) \subseteq F \subseteq E$, *there is a long exact sequence*

$$\dots \to \pi_{n+1}(B, b_0) \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \dots$$

With some conditions, the Serre spectral sequence (associated to a fibration) gives an exact sequence in cohomology.

Proposition 3 ([Mc] Example 5.D)**.** *Let* $F \to E \to B$ *be a fibration with* B *path-connected and [the system of local coefficients on* B *induced by* F *is simple]. If* $H^i(B; R) = 0$, $0 < i < p$ *and* $H^j(F; R) = 0$, $0 < j < q$, *then there is an exact sequence*

$$\begin{aligned} 0 &\to H^1(B; R) \to H^1(E; R) \to H^1(F; R) \to H^2(B; R) \to \dots \\ &\to H^{p+q-1}(B; R) \to H^{p+q-1}(E; R) \to H^{p+q-1}(F; R). \end{aligned}$$

The topology of $O(n)$ (and a detour through Clifford algebras)

It's easy to see that $O(n)$ has two path components, $\pm SO(n)$. The long exact sequence in homotopy for the fibration

$$SO(n) \to SO(n+1) \to S^n$$

is

$$\ldots \to \pi_k S^n \to \pi_k SO(n+1) \to \pi_k SO(n) \to \pi_{k+1} S^n \to \ldots$$

For $n \geq 3$, we get isomorphisms $\pi_1 SO(n+1) \cong \pi_1 SO(n)$ since $\pi_1 S^n$ and $\pi_2 S^n$ are trivial. It's also straightforward to see that $\pi_1 SO(3) \cong \mathbb{Z}/(2)$, since $SO(3) \cong \mathbb{RP}^3$. [Think of a rotation as a vector in $\mathbb{R}^3$ with length in $[-\pi, \pi]$, the length being the angle of rotation, and the vector defining an axis and orientation. The open ball of radius π consists of distinct rotations and and rotating by π and $-\pi$ around the same axis with opposite orientations gives the same rotation, identifying antipodal points of the sphere of radius π. Hence real projective three-space.] Therefore, for $n \geq 3$, $\pi_1 SO(n) \cong \mathbb{Z}/(2)$.

From the above we see that the universal covering space/group of $SO(n)$ is a double cover. This group is known as $Spin(n)$ and can be concretely realized as follows. Let $Cl(n)$ be the Clifford algebra for the quadratic space $(\mathbb{R}^n, q(x) = -\sum_{i=1}^n x_i^2)$. This algebra is 2^n dimensional with basis

$$\{e_I = e_{i_1} \cdot \ldots \cdot e_{i_k} : I = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\},\ i_1 < i_2 < \ldots < i_k\}$$

and relations generated by

$$e_i^2 = -1,\ e_i e_j + e_j e_i = 0, i \neq j,$$

where $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^n$. The space $\mathbb{R}^n = \text{span}_{\mathbb{R}}\{e_i\}$ sits in degree one and its non-zero elements are invertible, $x^{-1} = -x/q(x)$. The group Γ generated by vectors acts on $\mathbb{R}^n \subseteq Cl(n)$ via the "twisted adjoint representation"

$$\Gamma \to O(n),\ g \mapsto [v \mapsto g v g'^{-1}],$$

where $' : Cl(n) \to Cl(n)$ is the grade involution generated by $e_i' = -e_i$. For $g = x \in \mathbb{R}^n$, it can easily be checked that the map $v \mapsto x v x'^{-1}$ is reflection through the hyperplane perpendicular to x. The group generated by even products of vectors of norm one is $Spin(n)$, fitting into the exact sequence

$$1 \to \pm 1 \to Spin(n) \to SO(n) \to 1.$$

$\check{H}^1$ and principal bundles

Let B be a topological space, $\mathfrak{U} = \{U_i\}$ an open cover of B, and $\mathcal{F}$ a sheaf of Abelian groups on B.

The **Cech complex** (C^n, d^n)associated to the cover $\mathfrak{U}$ and the sheaf $\mathcal{F}$ is the complex of Abelian groups

$$0 \to \prod_i \Gamma(U_i, \mathcal{F}) \xrightarrow{d^0} \prod_{i<j} \Gamma(U_i \cap U_j, \mathcal{F}) \xrightarrow{d^1} \prod_{i<j<k} \Gamma(U_i \cap U_j \cap U_k, \mathcal{F}) \xrightarrow{d^2} \ldots$$

with differentials given by the alternating sum of restriction maps

$$(d^n \sigma)_{i_0, i_1, \ldots, i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \sigma_{i_1, \ldots, \widehat{i_k}, \ldots, i_{n+1}}.$$

If G is an Abelian group and $\xi : E \to B$ a principal G-bundle, we can consider its sheaf of sections. If $E = \cup_i \xi^{-1}(U_i)$ is a trivialization with isomorphisms $\Gamma(U_i, E) \xrightarrow{\phi_i} \Gamma(U_i, G)$ and transition maps $\phi_j \phi_i^{-1} = g_{ij} : U_i \cap U_j \to G$, then the transition maps g_{ij} satisfy the cocycle condition

$$1 = g_{jk} g_{ki} g_{ij} = \phi_k \phi_j^{-1} \phi_i \phi_k^{-1} \phi_j \phi_i^{-1},$$

i.e. the collection $\{g_{ij}\}$ is a Cech 1-cocycle (in the kernel of d^1). A Cech 1-coboundary $u_j u_i^{-1}$ is just a change of the trivialization, i.e. replacing ϕ_i with $u_i \phi_i$ with $u_i \in \Gamma(U_i, G)$. In other words, a Cech 1-cocycle is the data needed to glue together trivial bundles to form a principal bundle.

If G is not Abelian, most of the Cech complex gets destroyed (the differentials aren't homomorphisms, etc.) but $\check{H}^1$ still makes sense as a pointed set, defined as the quotient of the 1-cocycles $g_{jk} g_{ik}^{-1} g_{ij}$ modulo the action of 0-cochains induced by $\{g_i\}.\{g_{ij}\}_{ij} = \{g_j g_{ij} g_i^{-1}\}_{ij}$, and this equivalence relation still parameterizes principal bundles.

To summarize, for a space X we have

$$Prin_G(X) \cong \check{H}^1(X; G).$$

Orientation and w_1

Let M be a connected Riemannian n-manifold, and let $\xi : E \to M$ be its orthonormal frame bundle, a principal $O(n)$-bundle. This gives a fibration $O(n) \to E \to M$ and an exact sequence (reduced cohomology)

$$0 \to H^0(M; \mathbb{F}_2) \to H^0(E; \mathbb{F}_2) \to H^0(O(n); \mathbb{F}_2) \xrightarrow{w_1} H^1(M; \mathbb{F}_2).$$

The group H^0 is $\mathbb{F}_2^{r-1}$ where r is the number of connected components. M is connected and $O(n)$ has two components $\pm SO(n)$. There are two possibilities for $H^0(E)$, either 0 or $\mathbb{F}_2$. Hence the diagram is

$$0 \to 0 \to \{0 \text{ or } \mathbb{F}_2\} \to \mathbb{F}_2 \xrightarrow{w_1} H^1(M; \mathbb{F}_2).$$

If $H^0(E) = 0$, then M is not orientable (there is no $SO(n)$-valued section), and if $H^0(E)$ has rank one, then M is orientable. So M is orientable if and only if the connecting homomorphism above (labeled suggestively as w_1) is trivial.

Question: How does this relate to the definition of w_1 given above (or any other definition)?

Spin structures and w_2

Let M be an orientable connected Riemannian n-manifold, and let $\xi : E \to M$ be its oriented orthonormal frame bundle, a principal $SO(n)$-bundle. This gives a fibration $SO(n) \to E \to M$ and an exact sequence

$$0 \xrightarrow{w_1} H^1(M; \mathbb{F}_2) \to H^1(E; \mathbb{F}_2) \to H^1(SO(n); \mathbb{F}_2) \xrightarrow{w_2} H^2(M; \mathbb{F}_2).$$

Note that $H^1(X; G)$ is the space of isomorphism classes of principal G-bundles (clear via Cech cohomology since Cech 1-cocycles give gluing data).

A **spin structure** on M is a double-cover S of the oriented orthonormal frame bundle E such that the restriction to the fibers over points of M is the spin double cover $Spin(n) \to SO(n)$. From the above, a spin structure gives an element of $H^1(E; \mathbb{F}_2)$.

The group $H^1(SO(n); \mathbb{F}_2)$ is $\mathbb{F}_2$ since there are two double covers of $SO(n)$; the trivial double cover and the spin double cover. For a spin structure to exist, the map $H^1(E) \to H^1(SO(n))$ must be surjective, and therefore the connecting homomorphism (suggestively labeled w_2) be zero.

So, spin structures exist if and only if the connecting homorphism is zero, and the set of spin structures is in bijection with the kernel $H^1(M)$.

Question: How does this relate to the definition of w_2 given above (or any other definition)?

Another view

The short exact sequence

$$1 \to \{\pm 1\} \to Spin(n) \to SO(n) \to 1$$

gives an exact sequence in cohomology

$$H^0(M; SO(n)) \xrightarrow{\delta^0} H^1(M; \mathbb{Z}/2) \to H^1(M; Spin(n)) \to H^1(M; SO(n)) \xrightarrow{w_2} H^2(M; \mathbb{Z}/2).$$

The image of the orthnormal frame bundle under w_2 above is the second Stiefel-Whitney class. Also note that different spin structures may be isomorphim as abstract $Spin(n)$-bundles, the former being parameterized by $H^1(M, \mathbb{Z}/2)$ and the latter by $H^1(M, \mathbb{Z}/2)/\delta^0 H^0(M; SO(n))$.

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