Ergodic Theory of Simple Continued Fractions

Robert Hines

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1 Simple Continued Fractions

Every irrational number $x \in \mathbb{R}\backslash\mathbb{Q}$ has a unique representation of the form

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, \dots, a_n, \dots], a_0 \in \mathbb{Z}, a_i \in \{1, 2, 3, \dots\} i \geq 1$$

e.g.

$$\begin{aligned} \pi &= [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, \dots] \text{ (random?)}, \\ e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots] \text{ (not random)}, \\ \gamma &= [0; 1, 1, 2, 1, 2, 1, 4, 3, 13, \dots] \text{ (random?)}, \\ \frac{1+\sqrt{5}}{2} &= [1; 1, 1, 1, 1, 1, \dots] \text{ (not random)}. \end{aligned}$$

Rationals have two such (finite) representations

$$x = [a_0; a_1, \dots, a_n] = [a_0; a_1, \dots, a_n - 1, 1].$$

For rational x, the continued fraction expansion is essentially the euclidean algorithm, $(p, q) \mapsto (q, p \mod q)$, where we retain the quotient at each step. For instance

$$(355, 113) \overset{3}{\mapsto} (113, 16) \overset{7}{\mapsto} (16, 1) \overset{16}{\mapsto} (1, 0)$$

and

$$\frac{355}{113} = 3 + \frac{1}{7 + \frac{1}{16}}.$$

The a_i are obtained by

$$x_0 = x, a_0 = \lfloor x_0 \rfloor, x_{i+1} = \frac{1}{x_i - a_i} = [a_{i+1}; a_{i+2}, \dots], a_{i+1} = \lfloor x_{i+1} \rfloor.$$

If $x = [a_0; a_1, a_2, \dots]$ then the rational numbers

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$$

are the rational *convergents* of x. The convergents satisfy

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \cdots \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

which is the same as

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \cdots \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is the euclidean algorthim; if $a = bq + r$ then

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ r \end{pmatrix}.$$

This gives the recurrence relation

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}. \tag{1}$$

Taking determinants, we have

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n, \ \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}} \tag{2}$$

and a little algebra gives

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1} a_n}{q_n q_{n-2}}.$$

From this we see that the convergents with n even are increasing and the convergents with n odd are decreasing, and that each convergent with even n is less than each convergent with odd n. Hence the convergents with n even increase to some limit x^* and the convergents with n odd decrease to some limit x_* with $x^* \leq x_*$. The limits x^* and x_* are equal by (1) (show $q_n \geq 2^{(n-1)/2}$) and (2), proving the convergence of infinite simple continued fractions.

Also note

$$x = \lim_{n\to\infty} \frac{p_n}{q_n} = a_0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{q_k q_{k+1}}$$

and

$$\frac{1}{q_{n+2}} \leq |x q_n - p_n| \leq \frac{1}{q_{n+1}}.$$

One last identity that we will use is

$$x = \frac{p_n + p_{n-1} x_{n+1}}{p_n + p_{n-1} x_{n+1}}$$

(where $x_{n+1} = [a_{n+1}; a_{n+2}, \ldots])$ which follows from

$$\begin{pmatrix} p_{n+k}(x) \\ q_{n+k}(x) \end{pmatrix} = \begin{pmatrix} p_n(x) & p_{n-1}(x) \\ q_n(x) & q_{n-1}(x) \end{pmatrix} \begin{pmatrix} p_{k-1}(x_{n+1}) & p_{k-2}(x_{n+1}) \\ q_{k-1}(x_{n+1}) & q_{k-2}(x_{n+1}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

after dividing and letting $k \to \infty$.

[Fun fact: The limit of the ratio of successive Fibonacci numbers approaches the golden ratio.]

One reason to consider simple continued fractions are that the convergents are optimal in the following sense.

Theorem. *Let $x = [a_0; a_1, a_2, \ldots] \in \mathbb{R}\backslash\mathbb{Q}$. If $0 < q \le q_n$ then if $p/q \neq p_n/q_n$*

$$|qx - p| > |q_n x - p_n|$$

In particular

$$\left| x - \frac{p}{q} \right| > \left| x - \frac{p_n}{q_n} \right|.$$

Conversely, if a/b is such that $|a - bx| < |p - qx|$ for all $0 < q \le b, a/b \neq p/q$, then a/b is one of the convergents to x.

Proof. If $|qx - p| > |q_n x - p_n|$ and $0 < q < q_n$ then dividing by qq_n gives

$$\frac{1}{q}\left| \frac{p_n}{q_n} - x \right| < \frac{1}{q_n}\left| \frac{p}{q} - x \right| < \frac{1}{q}\left| \frac{p}{q} - x \right|$$

so that $\left| x - \frac{p}{q} \right| > \left| x - \frac{p_n}{q_n} \right|$.

To prove the first assertion, note that (because of the alternating nature of the convergents) we have

$$\left| x - \frac{p_n}{q_n} \right| = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| - \left| \frac{p_{n+1}}{q_{n+1}} - x \right|$$

so that

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1} q_{n+2}} = \frac{a_{n+2}}{q_n q_{n+2}}$$

and

$$\frac{1}{q_{n+2}} < |q_n x - p_n| < \frac{1}{q_{n+1}}.$$

Hence we may assume that $q_{n-1} < q \le q_n$. If $q = q_n$, then

$$\left| \frac{p}{q} - \frac{p_n}{q_n} \right| \ge \frac{1}{q_n}, \ \left| \frac{p_n}{q_n} - x \right| \le \frac{1}{q_n q_{n+1}} \le \frac{1}{2q_n}$$

and

$$\left|\frac{p}{q} - x\right| = \left|\frac{p}{q} - \frac{p_n}{q_n} + \frac{p_n}{q_n} - x\right| \geq \frac{1}{q_n} - \frac{1}{2q_n} = \frac{1}{2q}$$

proving $|qx - p| > |q_n x - p_n|$.

If $q_{n-1} < q < q_n$, let

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

with $a, b \in \mathbb{Z}$. Then $q = aq_n + bq_{n-1} < q_n$ and we must have $ab < 0$. We also know that $p_n - q_n x$ and $p_{n-1} - q_{n-1}x$ are of opposite sign as well, so that $a(p_n - q_n x)$ and $b(p_{n-1} - q_{n-1}x)$ have the same sign. Hence

$$p - qx = a(p_n - q_n x) + b(p_{n-1} - q_{n-1}x) \Rightarrow |p - qx| = |a(p_n - q_n x)| + |b(p_{n-1} - q_{n-1}x)|$$

and

$$|p - qx| > |p_{n-1} - q_{n-1}x| > |p_n - q_n x|$$

as desired.

Conversely, let $a/b \neq p_n/q_n$ for any n be a best approximant as in the statement of the theorem. If $a/b < a_0$ then

$$|x - a_0| < \left|x - \frac{a}{b}\right| \leq |bx - a| \ (b \geq 1),$$

a contradiction. Now, either $a/b > p_1/q_1$ or there is an n with a/b between p_{n-1}/q_{n-1} and p_{n+1}/q_{n+1}. In the first case, we again get a contradiction since

$$\left|x - \frac{a}{b}\right| > \left|\frac{p_1}{q_1} - \frac{a}{b}\right| \geq \frac{1}{bq_1}$$

implies $|bx - a| > 1/q_1 = 1/a_1$, but $|a_0 - x| \leq 1/(q_0 q_1) = 1/a_1$ and a_0 is a better approximation (with denominator 1). In the second case

$$\frac{p_{n-1}}{q_{n-1}} < \frac{a}{b} < \frac{p_{n+1}}{q_{n+1}} < x \text{ or } x < \frac{p_{n+1}}{q_{n+1}} < \frac{a}{b} < \frac{p_{n-1}}{q_{n-1}}$$

we have

$$\left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| \geq \frac{1}{bq_{n-1}}$$

and

$$\left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| \leq \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}}$$

so that $b > q_n$. On the other hand,

$$\left|x - \frac{a}{b}\right| \geq \left|\frac{p_{n+1}}{q_{n+1}} - \frac{a}{b}\right| \geq \frac{1}{bq_{n+1}}$$

so that $|bx - a| \geq 1/q_{n+1} \geq |q_n x - p_n|$. This is a contradiction since $q_n < b$ and $|bx - a| > |q_n x - p_n|$. □

One application of continued fractions is solving the Pell equation, $x^2 - Dy^2 = \pm 1$ ($D > 0$ squarefree), obtaining fundamental units in real quadratic fields. In particular, if $\sqrt{D} = [a_0; \overline{a_1, \ldots, a_s}]$ (periodic of period s) and $p/q = [a_0; \ldots a_{s-1}]$ then the fundamental unit is given by

$$\epsilon = p + q\sqrt{D},\ D \equiv 2, 3(4),\ D \equiv 1(8)$$

or one of

$$\epsilon = p + q\sqrt{D},\ \epsilon^3 = p + q\sqrt{D}$$

otherwise.

For example, with $D = 7$ we have $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$ so that $s = 2$, $p/q = [2; 1, 1, 1] = 8/3$ and $\epsilon = 8 + 3\sqrt{7}$ is a fundamental unit (i.e. $(\mathbb{Z}[\sqrt{7}])^\times = \pm\epsilon^{\mathbb{Z}}$).

2 Ergodic Theory

A *measure-preserving system* $(X, \mathcal{B}, \mu, T)$ is a finite measure space $(X, \mathcal{B}, \mu)$ equipped with a measureable $T : X \to X$ that is *measure-preserving* $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$. We say the system is *ergodic* if whenever $A \in \mathcal{B}$ satisfies $A = T^{-1}A$, then $\mu(A) \in \{0, \mu(X)\}$.

For (immediate) future use, we note that ergodicity is equivalent to

$$f \in L^2, f \circ T = f \Rightarrow f \text{ is constant a.e..}$$

Some examples:

1. Consider $T_b : [0, 1) \to [0, 1), T_b x = x + b \mod 1$. Then T_b preserves lebesgue measure (Haar measure). If $b = p/q$ is rational, then the system is not ergodic (if $A \subseteq (0, 1/q)$ then $\cup_{i=1}^q (A + i/q)$ is T_b-invariant). If b is irrational, then T_b is ergodic since if $f(x) = \sum_n a_n e^{2\pi inx}$ is T_b invariant, then $f(x) = f(x+b) = \sum_n a_n e^{2\pi ib} e^{2\pi inx}$ and $a_n(e^{2\pi ib} - 1) = 0$ for all $n \neq 0$. Since b is irrational, this is only possible if $a_n = 0$ for all $n \neq 0$.

2. Another example on the interval/circle is $T_k : [0, 1) \to [0, 1), T_k x = kx,\ k \in \mathbb{Z}\backslash\{0, 1\}$. This also preserves lebesgue measure (Haar measure). [In general, if $T : G \to G$ is a continuous endomorphism of a compact group, then T preserves Haar measure μ as follows. Let ν be the pushforward of μ by T, $\nu(E) = \mu(T^{-1}E)$. Then
$$\nu(TxE) = \mu(T^{-1}(TxE)) = \mu(xT^{-1}E) = \mu(T^{-1}E) = \nu(E).$$
Because T is surjective, ν is G-invariant and must be Haar measure, $\nu = \mu$.] T_k is ergodic since if $f \circ T = f$ with $f(x) = \sum_n a_n e^{2\pi inx}$ then for all j we have $f(x) = f(k^j x) = \sum_n a_n e^{2\pi ink^j x}$. Hence $a_n = a_{k^j n}$ and letting $j \to \infty$ (Riemann-Lebesgue: $\int_0^1 f(x)e^{2\pi inx}dx \to 0$) shows that $a_n = 0$ for all $n \neq 0$. Thus f is constant.

3. One more example. Let I be the incidence matrix of a digraph on n vertices, and suppose P be a stochastic matrix compatible with I ($I(i, j) = 0 \Rightarrow P(i, j) =$

0). Define a measure on the subset $X \subseteq \{1, \dots, n\}^{\mathbb{N}}$ where $x = (x_i) \in X$ iff $I(x_i, x_{i+1}) = 1$ for all i. Define a measure μ on the cylinder sets $U(y_1, \dots, y_k) = \{x \in X : x_1 = y_1, \dots, x_k = y_k\}$ by $\mu(U(y_1, \dots, y_k)) = \pi_{y_1} P(y_1, y_2) \dots P(y_{k-1}, y_k)$ where π is a stationary distribution (left eigenvector) for P. Then the left shift $T(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$ is measure preserving and T is ergodic iff P is irreducible.

The big theorem we will be using later is the following.

Theorem (Birkhoff Pointwise Ergodic Theorem, 1931)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. For any integrable $f : X \to \mathbb{C}$, the time average*

$$f^*(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

exists for a.e. $x \in X$. The time average f^* is T-invariant, $f^* \in L^1$, and $\int f d\mu = \int f^* d\mu$. If T is ergodic with respect to μ, then the time average is constant and equal to the space average

$$f^*(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \frac{1}{\mu(X)} \int_X f d\mu$$

for a.e. $x \in X$.

As you may imagine, this is a somewhat technical result. We will use the following.

Proposition (Maximal Inequality)**.** *Let $U : L^1(X) \to L^1(X)$ be positive ($f \geq 0 \Rightarrow Uf \geq 0$) with $\|U\| \leq 1$ and let $f \in L^1$ be real valued. If $f_0 = 0, f_n = \sum_{i=0}^{n-1} U^i f$ for $n \geq 1$ and $F_N(x) = \max\{f_n(x) : 0 \leq n \leq N\}$ (pointwise maximum), then*

$$\int_{\{F_N > 0\}} f d\mu \geq 0$$

for all N.

Proof. We have $F_N \in L^1$, $F_N \geq f_n$ for all n so that $UF_N \geq Uf_n$ for all n by positivity. Hence $UF_N + f \geq Uf_n + f = f_{n+1}$ and therefore

$$\begin{aligned} UF_N(x) + f(x) &\geq \max_{1\leq n\leq N} f_n \\ &= \max_{1\leq n\leq N} f_n \text{ when } F_N(x) \geq 0 \\ &= F_N(x). \end{aligned}$$

Thus $f \geq F_N - UF_N$ on $A = \{F_N > 0\}$ so that

$$\begin{aligned} \int_A f &\geq \int_A F_N - \int_A UF_N \\ &= \int_X F_N - \int_A UF_N \text{ since } F_N = 0 \text{ on } X \backslash A \\ &\geq \int_X F_N - \int_X UF_N \text{ since } F_N \geq 0 \Rightarrow UF_N \geq 0 \\ &\geq 0 \text{ since } \|U\| \leq 1. \end{aligned}$$

$\square$

Corollary. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and $g \in L^1$ real-valued. If $A \in \mathcal{B}$ is T-invariant, then*

$$\int_{B_\alpha \cap A} g d\mu \geq \alpha \mu(B_\alpha \cap A)$$

where

$$B_\alpha = \left\{ x : \sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} g(T^i x) > \alpha \right\} \right\}$$

Proof. We consider $T : A \to A$ and use the above with $Uh = h \circ T =, f = g - \alpha$. Then we have (in the notation above)

$$f_n(x) = \sum_{i=0}^{n-1} \left(g(T^i x) - \alpha \right), \ f_n(x) > 0 \iff \frac{1}{n} \sum_{i=0}^{n-1} g(T^i x) > \alpha$$

so that

$$x \in B_\alpha \iff x \in \{F_N > 0\} \text{ for some } N, \quad \text{i.e. } B_\alpha = \cup_N \{F_N > 0\}.$$

By the maximal inequality, we have

$$\int_{E_\alpha} f d\mu \geq 0, \ \int_{E_\alpha} g d\mu \geq \alpha \mu(E_\alpha).$$

□

(Proof of the pointwise ergodic theorem). idontwanna □

3 Continued Fractions as a Dynamical System

Consider the system

$$X = [0,1] \backslash \mathbb{Q}, \ T(x) = \left\{ \frac{1}{x} \right\} := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

In terms of the continued fraction expansion $x = [a_1, a_2, \dots]$, we have $T(x) = [a_2, a_3, \dots]$, i.e. T is the shift map on $\mathbb{N}^{\mathbb{N}}$. Gauss discovered (somehow) the following T-invariant probability measure (absolutely continuous w.r.t. lebesgue measure)

$$d\mu(x) = \frac{1}{\log 2} \frac{dx}{1+x}.$$

It's easy to verify that the Gauss measure is shift invariant. We check this on sets of the form $A = (0, a)$ (which generate the Borel sigma algebra)

$$
\begin{aligned}
(\log 2)\mu(T^{-1}(A)) &= \mu\left(\coprod_n \left(\frac{1}{n+a}, \frac{1}{n}\right)\right) \\
&= \sum_n \int_{\frac{1}{n+a}}^{\frac{1}{n}} \frac{dx}{1+x} = \sum_n \log\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+a}}\right) \\
&= \sum_n \log(n+1) - \log n - \log(n+a+1) + \log(n+a) \\
&= \log(1+a) + \lim_{N\to\infty} \log\left(\frac{N+1}{N+a+1}\right) \\
&= \log(1+a) = \int_0^a \frac{dx}{1+x} = (\log 2)\mu(A).
\end{aligned}
$$

Fun fact:

$$
\begin{aligned}
\int_0^1 \left\{\frac{1}{x}\right\} dx &= \sum_n \int_{1/(n+1)}^{1/n} \left(\frac{1}{x} - n\right) dx \\
&= \lim_{N\to\infty} \log(N+1) - \sum_{n=1}^{N} \frac{1}{n+1} = 1 - \gamma.
\end{aligned}
$$

4 Ergodicity of the Gauss Map

There are various proofs of ergodicity of the Gauss map. Perhaps the most interesting is viewing the Gauss map as a factor of a cross section of the geodesic flow on the unit tangent bundle of the modular surface $\mathcal{H}/PSL(2,\mathbb{Z})$. Another approach (a dynamical system on a space of quadratic forms) that may have been available to Gauss is outlined in Keane. For the sake of time here is a direct approach.

Proposition. *The measure preserving system*

$$X = [0,1]\backslash\mathbb{Q},\ T(x) = \{1/x\},\ d\mu = \frac{dx}{(1+x)\log 2}$$

is ergodic.

Proof. Consider the cylinder set

$$I(a_1, \ldots, a_n) = \{x = [a_1, \ldots, a_n, \ldots]\}$$

which is an interval in $(0, 1)$, either

$$([a_1, \ldots, a_n], [a_1, \ldots, a_n + 1]) \text{ or } ([a_1, \ldots, a_n + 1], [a_1, \ldots, a_n])$$

depending on whether n is even or odd. We want to show that

$$\mu\left(T^{-n}A\cap I(a_1,\ldots,a_n)\right)\asymp\mu(T^{-n}A)\mu(I(a_1,\ldots,a_n)) \tag{3}$$

for all Borel sets A, which will imply (since the sets $I(a_1,\ldots,a_n)$ generate the topology on $\mathbb{N}^{\mathbb{N}}$) that $\mu(A\cap B)\asymp\mu(A)\mu(B)$ for all B and any T-invariant A. Applying this to $B=(0,1)\backslash A$ gives $\mu(A)\in\{0,1\}$ as desired. To this end, we show (3) for intervals $A=[d,e]$.

Recall that

$$x=\frac{p_n+p_{n-1}T^nx}{p_n+p_{n-1}T^nx} \tag{4}$$

so that $x\in I(a_1,\ldots,a_n)\cap T^{-n}A$ if and only if x is as in (4) with $T^nx\in A=[d,e]$. Since T^n is monotone on $I(a_,\ldots,a_n)$, increasing for n even, decreasing for n odd,

$$\frac{p_n+\beta p_{n-1}}{q_n+\beta q_{n-1}}-\frac{p_n+\alpha p_{n-1}}{q_n+\alpha q_{n-1}}=(\beta-\alpha)\frac{q_np_{n-1}-p_nq_{n-1}}{(q_n+\beta q_{n-1})(q_n+\alpha q_{n-1})}$$
$$=(\beta-\alpha)\frac{(-1)^n}{(q_n+\beta q_{n-1})(q_n+\alpha q_{n-1})},$$

$I(a_1,\ldots,a_n)\cap T^{-n}A$ is an interval with endpoints

$$\frac{p_n+dp_{n-1}}{q_n+dq_{n-1}},\ \frac{p_n+ep_{n-1}}{q_n+eq_{n-1}}$$

and lebesgue measure (as above)

$$\frac{1}{(q_n+dq_{n-1})(q_n+eq_{n-1})}.$$

The lebesgue measure of $I(a_1,\ldots,a_n)$ is

$$\left|\frac{p_n}{q_n}-\frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right|=\frac{1}{q_n(q_n+q_{n+1})},$$

so that

$$ohgodidontwanna$$

□

5 Applications

Direct application of the ergodic theorem gives information about the continued fraction expansion of almost every number. Here are some examples.

Proposition. *For a.e.* $x = [a_1, a_2, a_3, \dots] \in [0,1]\backslash\mathbb{Q}$ *we have*

$$\mathbb{P}(a_n = k) = \lim_{N\to\infty} \frac{1}{N}|\{a_i = k, i \le N\}| = \frac{1}{\log 2}\log\left(\frac{(k+1)^2}{k(k+2)}\right),$$

$$(1 \sim 41.56\%, 2 \sim 16.99\%, 3 \sim 9.31\%, 4 \sim 5.89\%, \ \textit{etc.})$$

$$\lim_{N\to\infty}\left(\prod_{n=1}^{N} a_n\right)^{1/N} = \prod_k \left(\frac{(k+1)^2}{k(k+2)}\right)^{\log k/\log 2} = 2.6854520010...,$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N} a_n = \infty$$

$$\lim_{N\to\infty}\frac{1}{N}\log q_N = \frac{\pi^2}{12\log 2},$$

$$\lim_{N\to\infty}\frac{1}{N}\log\left|x - \frac{p_N}{q_N}\right| = -\frac{\pi^2}{6\log 2}.$$

Proof. Applying the ergodic theorem to the indicator $f = \mathbf{1}_{(1/(k+1),1/k)}$ gives the frequency/probability that a digit of the continued fraction expansion is given by k:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) = \lim_{N\to\infty}\frac{|\{i : a_i = k\}|}{N}$$
$$= \frac{1}{\log 2}\int_{1/(k+1)}^{1/k}\frac{dx}{1+x} = \frac{1}{\log 2}\log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

Applying the ergodic theorem to $f(x) = \sum_k (\log k)\mathbf{1}_{(1/(k+1),1/k)}$ we get

$$\int_{(0,1)} f\,d\mu = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log a_n$$
$$= \frac{1}{\log 2}\sum_k \int_{1/(k+1)}^{1/k}\frac{\log k}{1+x}dx$$
$$= \sum_k \frac{\log k}{\log 2}\log\left(\frac{(k+1)^2}{k(k+2)}\right)$$

so that, after exponentiating, we get

$$\lim_{N\to\infty}\left(\prod_{n=1}^{N} a_n\right)^{1/N} = \prod_k \left(\frac{(k+1)^2}{k(k+2)}\right)^{\log k/\log 2} = 2.6854520010...$$

(called Khinchin's constant, it is unknown if this constant is rational).

Applying the ergodic theorem to $f_M(x) = \sum_{k\le M} k\mathbf{1}_{(1/(k+1),1/k)}$, we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\le N \\ a_n \le M}} a_n = \sum_{k\le M} \frac{1}{\log 2} \int_{1/(k+1)}^{1/k} \frac{k}{1+x} dx$$
$$= \sum_{k\le M} k \log\left(\frac{(k+1)^2}{k(k+2)}\right) = \sum_{k\le M} k \log\left(1 + \frac{1}{k(k+2)}\right)$$
$$\ge \sum_{k\le M} \frac{1}{k+2} - \frac{1}{k(k+2)^2} \to \infty, \;\; M \to \infty.$$

With a bit more work we can also obtain results about the rate of convergence $[a_1, \ldots, a_n] \to x$, namely

$$\frac{1}{N} \log q_N \to \frac{\pi^2}{12 \log 2}, \;\; \frac{1}{N} \log \left| x - \frac{p_N}{q_N} \right| \to -\frac{\pi^2}{6 \log 2}.$$

To this end, recall from the first section that

$$x = \frac{p_n + p_{n-1} T^n x}{q_n + q_{n-1} T^n x},$$

from which it follows that

$$T^n x = -\frac{x q_n - p_n}{x q_{n-1} - p_{n-1}},$$
$$\prod_{i=0}^{n-1} T^i x = (-1)^n (x q_{n-1} - p_{n-1}) = |x q_{n-1} - p_{n-1}|,$$
$$x q_{n-1} - p_{n-1} = \frac{(-1)^{n+1}}{q_n + q_{n-1} T^n x}, \;\; |x q_{n-1} - p_{n-1}| \ge \frac{1}{2 q_n}.$$

Hence we have

$$\frac{1}{2q_n} \le |x q_{n-1} - p_{n-1}| \le \frac{1}{q_n} \text{ (or recall } \frac{1}{q_{n+1}} \le |x q_{n-1} - p_{n-1}| \le \frac{1}{q_n} \text{ from section 1)}$$

and

$$\frac{1}{2q_n} \le \prod_{i=0}^{n-1} T^i x \le \frac{1}{q_n}.$$

Taking logarithms and applying the ergodic theorem gives

$$\lim_{n\to\infty} \frac{1}{n} \log q_n = -\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(T^i x) = -\frac{1}{\log 2} \int_0^1 \frac{\log x}{1+x} dx,$$

the last integral being

$$-\frac{1}{\log 2}\int_0^1 \frac{\log x}{1+x}dx = \frac{1}{\log 2}\sum_{k=0}^{\infty}(-1)^{k+1}\int_0^1 x^k \log x dx$$
$$= \frac{1}{\log 2}\sum_{k=0}^{\infty}(-1)^{k+1}\left(\frac{x^{k+1}\log x}{k+1}\Big|_0^1 - \int_0^1 \frac{x^k}{k+1}dx\right)$$
$$= \frac{1}{\log 2}\sum_k \frac{(-1)^{k+1}}{k^2} = \frac{\zeta(2)}{2\log 2} = \frac{\pi^2}{12\log 2}$$

since

$$\sum_{k \text{ odd}} \frac{1}{k^2} - \sum_{k \text{ even}} \frac{1}{k^2} = \left(\zeta(2) - \sum_{k \text{ even}} \frac{1}{k^2}\right) - \sum_{k \text{ even}} \frac{1}{k^2}$$
$$= \zeta(2) - \frac{\zeta(2)}{4} - \frac{\zeta(2)}{4} = \frac{\zeta(2)}{2}.$$

Finally, because

$$\frac{1}{q_n q_{n+2}} \leq \left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}}$$

(recall

$$\left|x - \frac{p_n}{q_n}\right| \geq \left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \frac{a_{n+2}}{q_{n+2}q_n} \geq \frac{1}{q_{n+2}q_n}$$

from the first section) we have

$$-\frac{1}{n}\log\left|x - \frac{p_n}{q_n}\right| \to \frac{\pi^2}{6\log 2}$$

as $n \to \infty$. □

One last result, on the distribution of the normalized error $\theta_n(x) = q_n|p_n - q_n x|$.

Theorem. *Let* $\theta_n(x) = q_n^2\left|x - \frac{p_n}{q_n}\right|$. *Then for (lebesgue) almost every* $x \in [0,1]$

$$\lim_{N\to\infty}\frac{1}{N}|\{n : \theta_n(x) \leq z\}| = f(z)$$

where

$$f(z) = \begin{cases} \frac{z}{\log 2} & 0 \leq z \leq 1/2 \\ \frac{1-z+\log(2z)}{\log 2} & 1/2 \leq z \leq 1 \end{cases}$$

Proof. This uses mixing properties of an extension of the gauss map. See Hensley and the references there. □

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