Contents

[Sections 1,2, and 3 are OK - the rest need work/reorganization]

1 Dirichlet Series and The Riemann Zeta Function

Throughout, $s = \sigma + it$ is a complex variable (following Riemann).

Definition. *The Riemann zeta function, $\zeta(s)$, is defined by*

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

for $\sigma > 1$.

Lemma (Summation by Parts)**.** *We have*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

where $A_n = \sum_{k \le n} a_k$. In particular, if $\sum_{n \ge 1} a_n b_n$ converges and $A_n b_n \to 0$ as $n \to \infty$ then

$$\sum_{n \ge 1} a_n b_n = \sum_{n \ge 1} A_n(b_n - b_{n+1}).$$

Another formulation: if $a(n)$ is a funciton on the integers, $A(x) = \sum_{n \le x} a(n)$, and f is C^1 on $[y, x]$ then

$$\sum_{y < n \le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt$$

Proof.

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q}(A_n - A_{n-1})b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1}b_n$$
$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1}b_p.$$

For the second forumulation, assume y is not an integer and let $N = \lceil y \rceil, M = \lfloor x \rfloor$. We have

$$\begin{aligned}\int_y^x A(t)f'(t)dt =& A(N-1)\int_y^N f'(t)dt + A(M)\int_M^x f'(t)dt + \sum_{n=N}^{M-1} A(n)\int_n^{n+1} f'(t)dt \\ =& \left[A(N-1)f(N) - A(M)f(M) + \sum_{n=N}^{M-1} A(n)(f(n+1)-f(n))\right] \\ &+ A(M)f(x) - A(N-1)f(y) \\ =& -\sum_{y<n\le x} a(n)f(n) + A(x)f(x) - A(y)f(y).\end{aligned}$$

If y is an integer, one easily checks that the result still holds. □

Yet another version that is useful.

Lemma (Euler-Maclurin Summation)**.** *Assume f is C^1 on $[a,b]$ and let $W(x) = x - \lfloor x \rfloor - 1/2$. Then*

$$\sum_{a<n\le b} f(n) = \int_a^b (f(x) + W(x)f'(x))dx + \frac{1}{2}(f(b) - f(a)).$$

Proof. The right-hand side is

$$\begin{aligned}&\int_a^b f(x)dx + \int_a^b xf'(x)dx - \int_a^b \lfloor x \rfloor f'(x)dx - \frac{1}{2}\int_a^b f'(x)dx + \frac{1}{2}(f(b) - f(a)) \\ &= bf(b) - af(a) - \int_a^b \lfloor x \rfloor f'(x)dx = bf(b) - af(a) - \sum_{n=a}^{b-1} n(f(n+1) - f(n)) \\ &= \sum_{a<n\le b} f(n).\end{aligned}$$

□

A few lemmas on Dirichlet series ($\sum_n a_n n^{-s}$).

Lemma. *If $f(s) = \sum_n a_n n^{-s}$ converges for $s = s_0$ then $f(s)$ converges on $\sigma > \sigma_0$ (uniformly on compacta).*

Proof. We have $\sum_n a_n n^{-s} = \sum_n a_n n^{-(s-s_0)} n^{-s_0}$. Let $A_k(s_0) = \sum_{n=1}^k a_n n^{-s_0}$ and sum the tail of the series by parts

$$\sum_{n=M}^{N} \frac{a_n}{n^{s_0}} \frac{1}{n^{s-s_0}} = \sum_{n=M}^{N-1} A_n \left(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}} \right) - \frac{A_N}{N^{s-s_0}} + \frac{A_{M-1}}{M^{s-s_0}}.$$

We have

$$\left| \frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}} \right| = \left| (s-s_0) \int_n^{n+1} \frac{1}{x^{s-s_0+1}} dx \right| \le \frac{|s-s_0|}{n^{\sigma-\sigma_0+1}}$$

so that the tails go to zero uniformly. □

Lemma. *If* $|A_N| = |\sum_{n=1}^n a_n| < CN^{\sigma_0}$ *then* $f(s) = \sum_n a_n n^{-s}$ *converges for* $\sigma > \sigma_0$.

Proof. Summation by parts again:

$$\begin{aligned}
\left| \sum_{n=M}^{N} a_n n^{-s} \right| &= \left| \sum_{n=M}^{N-1} A_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) + \frac{A_N}{N^s} - \frac{A_{M-1}}{M^s} \right| \\
&= \left| \sum_{n=M}^{N} s \int_n^{n+1} \frac{A_n}{x^s} dx + \frac{A_N}{N^s} - \frac{A_{M-1}}{M^s} \right| \\
&\le C \left(\sum_{n=M}^{N} |s| \int_n^{n+1} \frac{dx}{x^{\sigma-\sigma_0+1}} + \frac{1}{N^{\sigma-\sigma_0}} + \left(\frac{M-1}{M} \right)^{\sigma_0} \frac{1}{M^{\sigma-\sigma_0}} \right)
\end{aligned}$$

which goes to zero. □

Proposition. *The Riemann zeta function can be continued to* $\sigma > 0$ *with a simple pole at* $s = 1$, $\operatorname{Res}_{s=1} \zeta(s) = 1$.

Proof.

$$\begin{aligned}
\zeta(s) &= \sum_n \frac{1}{n^s} = \sum_n n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\
&= \sum_n ns \int_n^{n+1} x^{-s-1} dx = s \int_1^\infty \lfloor x \rfloor x^{-s-1} dx \\
&= s \int_1^\infty (x - \{x\}) x^{-s-1} dx = \frac{s}{s-1} - s \int_1^\infty \{x\} x^{-s-1} dx \\
&= \frac{1}{s-1} + 1 + s \int_1^\infty \{x\} x^{-s-1} dx
\end{aligned}$$

where the last integral converges for $\sigma > 0$.

We can do better by writing

$$\begin{aligned}
\zeta(s) &= s \int_1^\infty \lfloor x \rfloor x^{-s-1} dx = s \int_1^\infty (x - 1/2 - (\{x\} - 1/2)) x^{-s-1} dx \\
&= \frac{s}{s-1} - \frac{1}{2} - s \int_1^\infty W(x) x^{-s-1} dx \\
&= \frac{s}{s-1} - \frac{1}{2} + s(s+1) \int_1^\infty \left(\int_1^x W(y) dy \right) x^{-s-2} dx
\end{aligned}$$

where $W(x) = x - \lfloor x \rfloor - 1/2$ is the "sawtooth" function. Since $\int_1^x W(y)dy$ is bounded for all x, the last integral converges for $\sigma > -1$. This also shows that $\zeta(0) = -1/2$.

[Another way to get this continuation is by considering

$$\begin{aligned}\zeta(s) = 2\zeta(s) - \zeta(s) &= 2\sum_{k=1}^{\infty}(2k)^{-s} + 2\sum_{k=1}^{\infty}(2k-1)^{-s} - \zeta(s) \\ &= 2^{1-s}\zeta(s) - \sum_{n=1}^{\infty}(-1)^n n^{-s} \\ &\Rightarrow \zeta(s) = \frac{\sum_n (-1)^n n^{-s}}{2^{1-s} - 1}\end{aligned}$$

where the sum on the right-hand side converges for $\sigma > 0$, clear for s real and Dirichlet series converge on half-planes.] □

Proposition. *The Riemann zeta function does not vanish on $\sigma \geq 1$.*

Proof. For $\sigma > 1$ we have the Euler product, which is non-zero (any convergent product, $\prod_n (1 + a_n), \sum_n |a_n| < \infty, |a_n| < 1$ is non-zero). Along the line $\sigma = 1$, we use a continuity argument, starting with the identity

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0.$$

Taking the logarithm of the Euler product, for $\sigma > 1$ we have

$$\Re \log \zeta(s) = \Re \sum_{m,p} \frac{1}{m} e^{-m(\sigma + it)\log p} = \sum_{m,p} \frac{1}{m} e^{-\sigma m \log p} \cos(mt \log p).$$

Taking $\theta = mt \log p$ in the inequality above, multiplying by $e^{-\sigma m \log p}/m$, and summing over m, p gives

$$3 \log \zeta(\sigma) + 4\Re \log \zeta(\sigma + it) + \Re \log \zeta(\sigma + 2it) \geq 0,$$

and exponentiating gives

$$\zeta^3(\sigma)|\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1.$$

Now, if $\zeta(1 + it) = 0$, taking limits as $\sigma \to 1$ above gives a contradiction, the quadruple zero cancels the triple pole and $\zeta(1 + 2it)$ remains bounded. Hence there are no zeros on $\sigma = 1$ as claimed. □

2 Primes in Arithmetic Progressions

Proposition. *The sum of the reciprocals of the primes diverges,*

$$\sum_p 1/p = \infty$$

Proof. Taking the logarithm of the Euler product for $\zeta(s)$ ($s > 1$ real) we get

$$\log(\zeta(s)) = -\sum_p \log(1 - p^{-s}) = \sum_{n,p} \frac{1}{np^{ns}}$$

using the power series expansion

$$-\log(1 - z) = \sum_n \frac{z^n}{n}.$$

The sum over $n > 1$ converges

$$\sum_{n,p} \frac{1}{n} p^{-ns} = \sum_p p^{-s} + O(1)$$

since

$$\sum_{p,\ n \geq 2} \frac{1}{n} p^{-ns} < \sum_p \frac{p^{-2}}{1 - p^{-s}} < \sum_n n^{-2} < \infty.$$

□

More specifically, although we don't need it and the above is merely motivational, we have

Theorem (Mertens). $\sum_{p \leq x} 1/p = C + \log\log x + O(1/\log x)$ *with* $C =$?

Proof. Let $S(x) = \sum_{n \leq x} \log n = \log(\lfloor x \rfloor !) = x \log x - x + O(\log x)$. Then

$$S(x) = \sum_{lm \leq x} \Lambda(l) = \sum_{l \leq x} \Lambda(l) \left\lfloor \frac{x}{l} \right\rfloor = x \sum_{l \leq x} \frac{\Lambda(l)}{l} + O(\psi(x)).$$

Since $\psi(x) \asymp x$ (see the prime number theorem section) we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

Since $\sum_{p, \alpha \geq 2} \Lambda(p)/p^\alpha < \infty$, we have

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Now use summation by parts

$$\begin{aligned}
\sum_{p \leq x} \frac{1}{p} &= \sum_{p \leq x} \frac{\log p}{p} \frac{1}{\log p} \\
&= \frac{1}{\log x} \sum_{p \leq x} \frac{\log p}{p} + \int_2^x \frac{\sum_{p \leq t} \frac{\log p}{p}}{t(\log t)^2} dt \\
&= 1 + O(1/\log x) + \int_2^x \frac{1}{t \log t} dt + O\left(\int_2^x \frac{1}{t(\log t)^2} \right) \\
&= C + \log(\log x) + O(1/\log x)
\end{aligned}$$

for some constant C. □

Definition. *A Dirichlet character to the modulus q, $\chi : \mathbb{Z} \to \mathbb{C}$, is induced by a homomorphism $(\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ and defined to be zero for $(n, q) > 1$. They form an abelian group under pointwise multiplication isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$ with identity χ_0 (the principal character) and $\chi^{-1} = \bar{\chi}$. Also note that Dirichlet characters are completely multiplicative, $\chi(ab) = \chi(a)\chi(b)$.*

Lemma (Orthogonality relations)**.**

$$\frac{1}{\phi(q)} \sum_{\chi} \chi(a) = \begin{cases} 1 & a \equiv 1(q) \\ 0 & \textit{else} \end{cases}$$

$$\frac{1}{\phi(q)} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) = \begin{cases} 1 & \chi = \chi_0 \\ 0 & \textit{else} \end{cases}$$

Definition. *The Dirichlet L-series associated to a character χ, $L(s, \chi)$, is defined by*

$$L(s, \chi) = \sum_n \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

for $\sigma > 1$.

For future reference we note that the series on the left actually converges on $\sigma > 0$ for non-principal χ and that $L(s, \chi_0)$ can be continued to $\sigma > 0$ with a simple pole at $s = 1$. Using summation by parts (cf. the section on the Riemann zeta function) we have, for non-principal χ

$$\sum_n \frac{\chi(n)}{n^s} = \sum_n \left(\sum_{k=1}^{n} \chi(k) \right) (n^{-s} - (n+1)^{-s}) = s \int_1^\infty \left(\sum_{n \leq x} \chi(n) \right) x^{-s-1} dx$$

with $\sum_{n \leq x} \chi(n) \leq \phi(q)$, whereas for χ_0 we have

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s})$$

and the claim follows from the properties of the Riemann zeta function.

Theorem (Primes in Arithmetic Progressions)**.** *For $(a, q) = 1$ there are infinitely many primes p such that $p \equiv a(q)$. More precisely, the sum of the reciprocals of such primes diverges, $\sum_{p \equiv a(q)} 1/p = \infty$.*

Proof. Taking the logarithm of the Euler product for $L(s, \chi)$ gives (similar to the above)

$$\log(L(s, \chi)) = -\sum_p \log(1 - \chi(p)p^{-s}) = \sum_{n,p} \frac{\chi(p)}{np^{ns}} = \sum_p \frac{\chi(p)}{p^s} + O(1).$$

Multiplying by $\bar{\chi}(a)/\phi(q)$ and summing over all characters moduluo q selects primes congruent to a modulo q (using orthogonality)

$$\frac{1}{\phi(q)}\sum_{\chi}\bar{\chi}(a)\log(L(s,\chi)) = \sum_{p} p^{-s}\sum_{\chi}\chi(pa^{-1}) + O(1) = \sum_{p\equiv a(q)}\frac{1}{p^s} + O(1).$$

On the left hand side, the term corresponding to the prinipal character diverges. If $L(1,\chi)\neq 0$ for the non-principal characters (we know that $L(1,\chi)$ is well-defined) then letting $s\to 1^+$ gives the divergence of the sum on the right-hand side,

$$\sum_{p\equiv a(q)}\frac{1}{p} = \infty$$

indicating the existence of infinitely many primes in a given arithmetic progression $a \bmod q$. □

To validate the proof above, we must prove the non-vanishing of $L(1,\chi)$ for non-principal characters. Three proofs are provided below.

Theorem (Non-vanishing of $L(1,\chi)$)**.** *For a non-principal character χ, we have*

$$L(1,\chi)\neq 0.$$

(Proof 1, de la Vallée Poussin). First note that for $s>1$ real, we have

$$F(s) = \prod_{\chi} L(s,\chi) \geq 1$$

since its logarithm is positive

$$\log(F(s)) = \sum_{\chi,p,n}\frac{\chi(p)}{np^{ns}} = \sum_{p\equiv 1(q),n}\frac{1}{np^{ns}} > 0.$$

If $L(1,\chi)=0$ for a complex character $\chi\neq\bar{\chi}$, then $L(1,\bar{\chi})=0$ as well. From this we see that $F(s)$ has a zero at $s=1$ (exactly one pole from the principal character, and at least two zeros from $\chi,\bar{\chi}$), a contradiction.

So we need only consider real characters. Given a real character χ with $L(1,\chi)=0$, consider the auxiliary function

$$\psi(s) = \frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}$$

which is analytic on $\sigma > 1/2$ with $\lim_{s\to 1/2^+}\psi(s) = 0$. Consider the product expansion for ψ

$$\begin{aligned}\psi(s) &= \prod_{p}(1-\chi(p)p^{-s})^{-1}(1-\chi_0(p)p^{-s})^{-1}(1-\chi_0(p)p^{-2s})\\ &= \prod_{p\nmid q}\frac{1-p^{-2s}}{(1-p^{-s})(1-\chi(p)p^{-s})} = \prod_{\chi(p)=1}\frac{1+p^{-s}}{1-p^{-s}}\\ &= \prod_{\chi(p)=1}\left(1+\sum_{n=1}^{\infty}2p^{-ns}\right).\end{aligned}$$

It follows that $\psi(s) = \sum_n a_n n^{-s}$ is a Dirichlet series with positive coefficients and $a_0 = 1$. Now expand ψ as a power series around $s = 2$, $\psi(s) = \sum_m b_m (s-2)^m$, and note that the radius of convergence is at least $3/2$ (the first singularity is at $s = 1/2$). The coefficients are given by

$$b_m = \frac{\psi^{(m)}(2)}{m!} = \frac{1}{m!} \sum_n a_n (-\log n)^m n^{-2} = (-1)^m c_m$$

for some non-negative c_m. Hence

$$\psi(s) = \sum_m c_m (2-s)^m$$

with $c_m \geq 0$ and $c_0 = \psi(2) = \sum_n a_n n^{-2} \geq a_0 = 1$. From this it follows that $\psi(s) \geq 1$ for $s \in (1/2, 2)$, contradicting $\psi(s) \to 0$ as $s \to 1/2^+$. Therefore $L(s,\chi) \neq 0$ as desired. □

(Proof 2, taken from Serre). We reconsider the function $F(s)$ from above, and claim an equality

$$F(s) = \prod_\chi L(s,\chi) = \prod_{p,\chi} (1 - \chi(p)p^{-s})^{-1} = \prod_{p \nmid q} (1 - p^{-f(p)s})^{-g(p)}$$

where $f(p)$ is the order of p modulo q and $g(p) = \phi(q)/f(p)$. [Note that $F(s)$ is the Dedekind zeta function of the qth cyclotomic field, away from the ramified primes.] By definition, $\chi(p)$ is an fth root of unity and for each choice of such a root, there are g choices to extend the character from the subgroup of $(\mathbb{Z}/q\mathbb{Z})^\times$ generated by p to the entire group. Hence

$$\prod_\chi (1 - \chi(p)T) = (1 - T^f)^g.$$

If $L(1,\chi) = 0$ for some non-principal χ, then $F(s)$ is analytic at for $\sigma > 0$ (the L-series for non-principal χ already are, and the simple pole of $L(s,\chi_0)$ at $s = 1$ is balanced by the supposed zero of $L(s,\chi)$ at $s = 1$). However, looking at the product expansion, for $s > 0$ we have

$$(1 - p^{-fs})^{-g} = \left(\sum_k p^{-kfs} \right)^g \geq \sum_k p^{-\phi(q)ks} = (1 - p^{-\phi(q)s})^{-1}$$

(taking diagonal terms and noting $fg = \phi(q)$) so that

$$F(s) \geq \prod_{p \nmid q} (1 - p^{-\phi(q)s})^{-1} = \zeta(\phi(q)s) \prod_{p \mid q} (1 - p^{-\phi(q)s})$$

which diverges at $s = 1/\phi(q)$, a contradiction. Therefore, there can be no χ with $L(1,\chi) = 0$. □

(Proof 3, Monsky). Here is an elementary proof for the non-vanishing of $L(1,\chi)$ for non-principal real χ. Let $c_n = \sum_{d \mid n} \chi(d)$. Note that $c_n \geq 0$ since c_n is multiplicative and

$$c_{p^a} = 1 + \chi(p) + \chi(p)^2 + \cdots + \chi(p)^a \geq 0.$$

Also note that $\sum_n c_n = \infty$ since $c_{p^a} = 1$ for any prime dividing q. Now consider the function (convergent on $[0,1)$)

$$f(t) = \sum_n \chi(n) \frac{t^n}{1-t^n} = \sum_n \sum_d \chi(n) t^{nd} = \sum_n t^n c_n$$

which we showed satisfies $f(t) \to \infty$ as $t \to 1^-$. If $\sum_n \chi(n)/n = 0$ then ($t \in [0,1)$)

$$-f(t) = \sum_n \left(\frac{\chi(n)}{n} \frac{1}{1-t} - \frac{\chi(n) t^n}{1-t^n} \right) = \sum_n \chi(n) \left(\frac{1}{n(1-t)} - \frac{t^n}{1-t^n} \right) =: \sum_n \chi(n) b_n.$$

Note that $\sum_{n \le x} \chi(n) \le \phi(q)$ and that $b_n \to 0$. If we can show that the b_n are decreasing, then the series converges for all $t \in [0,1)$ (summation by parts!) contradicting $f(t) \to \infty$ as $t \to 1^-$. To this end, we have

$$\begin{aligned}(1-t)(b_n - b_{n+1}) &= \frac{1}{n} - \frac{1}{n+1} - \frac{t^n}{1+t+\cdots+t^{n-1}} + \frac{t^{n+1}}{1+t+\cdots+t^n} \\ &= \frac{1}{n(n+1)} - \frac{t^n}{(1+t+\cdots+t^{n-1})(1+t+\cdots+t^n)}.\end{aligned}$$

By the arithmetic-geometric mean inequality, we have ($t \in [0,1)$)

$$\begin{aligned}\frac{1}{n}(1+t+\cdots+t^{n-1}) &\ge (t^{n(n-1)/2})^{1/n} \ge t^{n/2}, \\ \frac{1}{n+1}(1+t+\cdots+t^n) &\ge (t^{n(n+1)/2})^{1/(n+1)} \ge t^{n/2}.\end{aligned}$$

Hence

$$\begin{aligned}(1-t)(b_{n+1} - b_n) &= \frac{1}{n(n+1)} - \frac{t^n}{(1+t+\cdots+t^{n-1})(1+t+\cdots+t^n)} \\ &\ge \frac{1}{n(n+1)} - \frac{t^n}{t^n n(n+1)} = 0\end{aligned}$$

and the theorem follows. □

[Note: add some kind of density statement.]

3 Functional equations for $\zeta(s), L(s,\chi)$

Lemma (Poisson Summation). *Suppose f is "nice" and decays sufficiently fast at infinity, so that the periodization $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$ converges and is equal to its Fourier series. Then*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

where $\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t x} dt$.

Proof. We have

$$\sum_{n\in\mathbb{Z}} f(n) = F(0) = \sum_{n\in\mathbb{Z}} \left(\int_0^1 F(x)e^{-2\pi inx}dx\right) e^{2\pi in0} = \sum_{n\in\mathbb{Z}} \int_0^1 \left(\sum_{m\in\mathbb{Z}} f(x+m)\right) e^{-2\pi inx}dx$$

$$= \sum_{n,m\in\mathbb{Z}} \int_m^{m+1} f(x)e^{-2\pi inx}dx = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

□

Lemma. *The function*

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$$

satisfies the functional equation

$$\theta(1/x) = x^{1/2}\theta(x).$$

Proof. Apply Poisson summation to $f(z) = e^{-\pi z^2/x}$ with $x > 0$ fixed. We have

$$\hat{f}(n) = \int_{-\infty}^{\infty} e^{-\pi z^2/x}e^{-2\pi inz}dz \quad (z \mapsto x^{1/2}z)$$
$$= x^{1/2}\int_{-\infty}^{\infty} e^{-\pi(z^2+2inx^{1/2}z)}dz = x^{1/2}e^{-\pi n^2 x}\int_{-\infty}^{\infty} e^{-\pi(z+inx^{1/2})^2}dz$$
$$= x^{1/2}e^{-\pi n^2 x}$$

since the last integral is 1, comparing it to

$$\left(\int_{-\infty}^{\infty} e^{-\pi z^2}dz\right)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2}rdrd\theta = 1$$

say by integrating around a long rectangle along the real axis. [Or note that $g(z) = e^{-\pi z^2}$ is its own Fourier transform so that $\hat{f}(z) = g(z/\sqrt{x})^\wedge = \sqrt{x}\hat{g}(\sqrt{x}z) = \sqrt{x}g(\sqrt{x}z) = \sqrt{x}e^{-\pi z^2 x}$.] Summing over n we get

$$\theta(1/x) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) = x^{1/2}\theta(x).$$

□

Theorem (Functional equation for $\zeta(s)$)**.** *The equation*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty}(x^{s/2} + x^{(1-s)/2})\omega(x)\frac{dx}{x}$$

(where $\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} = (\theta(x)-1)/2$) gives a continuation of $\zeta(s)$ to the whole plane. The expression on the right is invariant under $s \leftrightarrow 1-s$, so that

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s),$$

and $\xi(s)$ is entire.

Proof. We start with the gamma function (say for $\sigma > 1$)

$$\Gamma(s/2) = \int_0^\infty e^{-x} x^{s/2-1} dx$$

and make a change of variable, $x = n^2\pi t$ to obtain

$$n^{-s}\pi^{-s/2}\Gamma(s/2) = \int_0^\infty e^{-n^2\pi t} t^{s/2-1} dt.$$

Sum over n and interchange limits ($\sum_n \int_0^\infty e^{-n^2\pi t} t^{s/2-1} dt$ converges uniformly for $\sigma > 0$) to obtain

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \omega(t) t^{s/2-1} dt.$$

Split the integral at $t = 1$ and use the functional equation for $\theta(t)$

$$\begin{aligned}\omega(1/t) &= \frac{1}{2}(\theta(1/t) - 1) = \frac{1}{2}\left(t^{1/2}\theta(t) - 1\right) \\ &= \frac{1}{2}\left(t^{1/2}(2\omega(t) + 1) - 1\right) = -\frac{1}{2} + \frac{t^{1/2}}{2} + t^{1/2}\omega(t)\end{aligned}$$

to obtain

$$\begin{aligned} n^{-s}\pi^{-s/2}\Gamma(s/2) &= \int_0^\infty \omega(t) t^{s/2-1} dt = \int_0^1 \omega(t) t^{s/2-1} dt + \int_1^\infty \omega(t) t^{s/2-1} dt \\ (t \mapsto 1/t) &= \int_1^\infty \left[\left(-\frac{1}{2} + \frac{t^{1/2}}{2} + t^{1/2}\omega(t)\right) t^{-s/2+1}\right] \frac{1}{t^2} dt + \int_1^\infty \omega(t) t^{s/2-1} dt \\ &= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left[\omega(t) t^{-s/2-1/2} + \omega(t) t^{s/2-1}\right] dt \\ &= \frac{1}{s(s-1)} + \int_1^\infty \left(t^{-s/2-1/2} + t^{s/2-1}\right) \omega(t) dt \\ &= \frac{1}{s(s-1)} + \int_1^\infty \left(t^{(1-s)/2} + t^{s/2}\right) \omega(t) \frac{dt}{t}. \end{aligned}$$

Note that the last integral converges for all s. □

Here is another version/proof of the functional equation for $\zeta(s)$.

Theorem. *The Riemann zeta function satisfies*

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(s-1) \sin(\pi s/2) \zeta(1-s).$$

Proof. Start with the gamma integral, make a change of variable $y = nx$

$$\Gamma(s) = \int_0^\infty y^{s-1} e^{-y} dy = n^s \int_0^\infty x^{s-1} e^{-nx} dx,$$

divide by n^s and sum over n to obtain

$$\Gamma(s)\zeta(s) = \sum_n \int_0^\infty x^{s-1} e^{-nx} dx = \int_0^\infty x^{s-1} \left(\sum_n e^{-nx} \right) dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Consider the contour integral

$$I(s) = \int_C \frac{z^{s-1}}{e^z - 1} dz$$

where C goes from $+\infty$ just above the real axis, circles the origin once (avoiding non-zero roots of $e^z - 1$), and returns to $+\infty$ just below the real axis. If C_ρ is the circle around the origin and $z = \rho e^{i\theta}$, then for $\sigma > 1$ we have

$$\left| \int_{C_\rho} \frac{z^{s-1}}{e^z - 1} dz \right| \leq 2\pi\rho \frac{\rho^{\sigma-1}}{|\sum_{n\geq 1} \rho^n e^{in\theta}/n!|} = \frac{2\pi\rho^{\sigma-1}}{|e^{i\theta} + \rho(\dots)|} \to 0 \text{ as } \rho \to 0.$$

Thus

$$\begin{aligned} I(s) &= -\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx + \int_0^\infty \frac{e^{2\pi i(s-1)} x^{s-1}}{e^x - 1} dx = (e^{2\pi i(s-1)} - 1) \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \\ &= (e^{2\pi i(s-1)} - 1)\Gamma(s)\zeta(s) = (e^{2\pi i(s-1)} - 1) \frac{\pi}{\sin(\pi s)\Gamma(1-s)} \zeta(s) \\ &= \zeta(s) \frac{2\pi i e^{\pi i s}}{\Gamma(1-s)}, \end{aligned}$$

(using the identity $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$) so that

$$\zeta(s) = e^{-i\pi s}\Gamma(1-s) \frac{1}{2\pi i} \int_C \frac{z^{s-1}}{e^z - 1} dz.$$

To get the functional equation, consider the contour C_n starting at $+\infty$ just above the real axis, going around the square defined by the vertices $\{(2n+1)\pi(\pm 1 \pm i)\}$, and returning to $+\infty$ just below the real axis. We use residues ($(e^z - 1)^{-1}$ has simple poles with residue 1 at $2\pi i k, k \in \mathbb{Z}$) to calculate

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_n - C} \frac{z^{s-1}}{e^z - 1} dz &= \sum_{k=1}^n [(2\pi i k)^{s-1} + (-2\pi i k)^{s-1}] = \sum_{k=1}^n (2\pi k)^{s-1} (i^{s-1} + (-i)^{s-1}) \\ &= e^{(s-1)\pi i} 2^s \pi^{s-1} \cos\left(\frac{\pi(s-1)}{2} \right) \sum_{k=1}^n k^{s-1} \text{ (note } -i = e^{3\pi i/2}) \\ &= e^{(s-1)\pi i} 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2} \right) \sum_{k=1}^n k^{s-1}. \end{aligned}$$

This gives us

$$\begin{aligned} I(s) = \int_C \frac{z^{s-1}}{e^z - 1} dz &= \int_{C_n} \frac{z^{s-1}}{e^z - 1} dz - \int_{C_n - C} \frac{z^{s-1}}{e^z - 1} dz \\ &= \int_{C_n} \frac{z^{s-1}}{e^z - 1} dz - i e^{(s-1)\pi i} 2^{s+1} \pi^s \sin\left(\frac{\pi s}{2} \right) \sum_{k=1}^n k^{s-1}. \end{aligned}$$

Note that $\int_{C_n} \frac{z^{s-1}}{e^z-1}dz \to 0$ as $n \to \infty$ to get the functional equation by taking the limit $n \to \infty$ above

$$\begin{aligned} I(s) = \zeta(s)\frac{2\pi i e^{\pi i s}}{\Gamma(1-s)} &= -ie^{(s-1)\pi i}2^{s+1}\pi^s \sin\left(\frac{\pi s}{2}\right)\zeta(1-s) \\ \Rightarrow \zeta(s) &= 2^s\pi^{s-1}\Gamma(s-1)\sin(\pi s/2)\zeta(1-s). \end{aligned}$$

□

An easy consequence of the above work is the value of $\zeta(s)$ at non-negative integers and at positive even integers. From the integral representation of $\zeta(s)$ above, we have, for $n \geq 0$,

$$\begin{aligned} \zeta(-n) &= e^{n\pi i}\Gamma(n+1)\frac{1}{2\pi i}\int_C \frac{z^{-n-1}}{e^z-1}dz \\ &= (-1)^n n!\frac{1}{2\pi i}\int_C z^{-n-2}\left(\sum_n \frac{B_n}{n!}z^n\right)dz \\ &= (-1)^n n!\frac{B_{n+1}}{(n+1)!} = (-1)^n\frac{B_{n+1}}{n+1} \end{aligned}$$

where the Bernoulli numbers, B_n come from the coefficients of the Taylor series

$$\frac{z}{e^z-1} = \sum_n \frac{B_n}{n!}z^n.$$

For instance

$$\zeta(0) = -\frac{1}{2},\ \zeta(-1) = -\frac{1}{12},\ \zeta(-3) = \frac{1}{120},\ \text{and } \zeta(-2n) = 0 \text{ for } n \geq 1.$$

Using the functional equation, we get, for $n \geq 1$ odd (else we get $0 = 0$ below)

$$\begin{aligned} (-1)^n\frac{B_{n+1}}{n+1} &= \zeta(-n) = 2^{-n}\pi^{-n-1}\sin(-n\pi/2)\zeta(1+n)\Gamma(1+n) \\ \Rightarrow \zeta(2m) &= \frac{B_{2m}}{(2m)!}2^{2m-1}\pi^{2m}(-1)^{m+1}. \end{aligned}$$

For instance

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945}.$$

We now move on to the functional equation for $L(s,\chi)$ where χ is *primitive*, i.e. q is the smallest period of χ in that there does not exist a divisor d of q and a character χ' modulo d such that χ is given by the composition

$$(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times \xrightarrow{\chi'} \mathbb{C}^\times.$$

Definition. *A Gauss sum $\tau(\chi)$ associated to a character χ of modulus q is*

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

More generally, define

$$\tau(\chi, z) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m z/q}.$$

Lemma. *If χ is primitive, we have*

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{m} \bar{\chi}(m) e^{2\pi i m n/q} = \frac{\tau(\bar{\chi}, n)}{\tau(\bar{\chi})},$$

and $|\tau(\chi)|^2 = q$.

Proof. If $(n, q) = 1$ we have

$$\chi(n)\tau(\bar{\chi}) = \chi(n) \sum_{m} \bar{\chi}(m) e^{2\pi i m/q} = \sum_{m} \bar{\chi}(mn^{-1}) e^{2\pi i m n^{-1} n/q} = \sum_{k} \bar{\chi}(k) e^{2\pi i k n/q} = \tau(\bar{\chi}, n)$$

where $k = mn^{-1}$ modulo q, whether or not χ is primitive. The last expression also holds for $(n, q) = d > 1$ if χ is primitive (in which case both sides are zero). It cannot be the case that χ is trivial on the kernel $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times$ else we extend from the image to a character $\chi' : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ and χ is not primitive. Hence there is a b prime to q and congruent to 1 modulo d with $\chi(b) \neq 1$, in which case $bn \equiv n(q)$ since

$$b - 1 = ld = l\frac{q}{(n, q)} \Rightarrow n(b-1) = \frac{nl}{(n, q)} q.$$

Therefore

$$\tau(\chi, n) = \sum_{a} \chi(a) e^{2\pi i a n/q} = \sum_{a} \chi(ab) e^{2\pi i a b n/q} = \chi(b) \sum_{a} \chi(a) e^{2\pi i a n/q} = \chi(b)\tau(\chi, n)$$

with $\chi(b) \neq 1$ so that $\tau(\chi, n) = 0$ as desired.

Finally, we show that for χ primitive, $|\tau(\chi)| = q^{1/2}$. Using the expression above we have

$$\begin{aligned}
\phi(q)|\tau(\bar{\chi})|^2 &= \sum_{n} |\chi(n)|^2 |\tau(\bar{\chi})|^2 = \sum_{n,k,l} \chi(k)\bar{\chi}(l) e^{2\pi i (l-k)n/q} \\
&= \sum_{k=l} \sum_{n} |\chi(k)|^2 + \sum_{k \neq l} \chi(k)\bar{\chi}(l) \sum_{n} e^{2\pi i (l-k)n/q} \\
&= q\phi(q) + 0,
\end{aligned}$$

and $|\tau(\chi)|^2 = q$ as desired. Hence we can divide by $\tau(\chi)$ and the lemma follows. □

We need functional equations similar to that of $\theta(1/x) = x^{1/2}\theta(x)$ used above.

Proposition (Functional equations for ψ_0, ψ_1)**.** *Given a primitive character χ of modulus q, define ψ_0, ψ_1 (for $\chi(-1) = 1$ or -1) by*

$$\psi_0(\chi, x) = \sum_{n\in\mathbb{Z}} \chi(n) e^{-n^2\pi x/q}, \ \psi_1(\chi, x) = \sum_{n\in\mathbb{Z}} n\chi(n) e^{-n^2\pi x/q}.$$

These functions satisfy the functional equations

$$\psi_0(\bar\chi, 1/x) = (x/q)^{1/2}\tau(\bar\chi)\psi_0(\chi, x), \ \psi_1(\bar\chi, 1/x) = -ix^{3/2}q^{-1/2}\tau(\bar\chi)\psi_1(\chi, x).$$

Proof. Define functions

$$f(z, x) = e^{-\pi z^2/x}, \ f_0(z, x) = e^{-\pi(qz+b)^2/(qx)}, \ f_1(z, x) = (qz + b)e^{-\pi(qz+b)^2/(qx)}$$

so that

$$f_0(z, x) = f(qz + b, qx), \ f_1(z, x) = \frac{qz}{-2\pi}\frac{\partial f}{\partial z}(qz + b, qx).$$

We showed earlier that $\hat f(z, x) = x^{1/2} f(z, 1/x)$ and applied Poisson summation to get the functional equation for θ earlier. We piggyback off of this using properties of the Fourier transform, namely

$$(g(z + b))^\wedge = e^{2\pi ib}\hat g(z), \ (g(az))^\wedge = \frac{1}{a}\hat g(z/a), \ \left(\frac{\partial g}{\partial z}\right)^\wedge = 2\pi iz\hat g(z).$$

Hence

$$\begin{aligned}
\hat f_0(z, x) &= q^{-1}e^{2\pi izb/q}\hat f(z/q, qx) = (x/q)^{1/2}e^{-\pi z^2 x/q}e^{2\pi izb/q}, \\
\hat f_1(z, x) &= \frac{qz}{-2\pi}\left(\frac{\partial f}{\partial z}(qz + b, qx)\right)^\wedge = \frac{x}{-2\pi}e^{2\pi izb/q}\left(\frac{\partial f}{\partial z}\right)^\wedge (z/q, qx) \\
&= -ix^{3/2}q^{-1/2}ze^{2\pi izb/q}e^{-\pi z^2 x/q}.
\end{aligned}$$

By Poisson summation, we have

$$\begin{aligned}
\sum_n \hat f_0(n, x) &= \sum_n (x/q)^{1/2}e^{-\pi n^2 x/q}e^{2\pi inb/q} = \sum_n f_0(n, x) = \sum_n e^{-\pi(qn+b)^2/(qx)}, \\
\sum_n \hat f_1(n, x) &= \sum_n -ix^{3/2}q^{-1/2}ne^{2\pi inb/q}e^{-\pi n^2 x/q} = \sum_n f_1(n, x) = \sum_n (qn + b)e^{-\pi(qn+b)^2/(qx)}.
\end{aligned}$$

Now multiply by $\bar\chi(b)$ and sum over b modulo q to get

$$\begin{aligned}
(x/q)^{1/2}\sum_n e^{-\pi n^2 x/q}\sum_b \bar\chi(b)e^{2\pi inb/q} &= \sum_{n,b}\bar\chi(b)e^{-\pi(qn+b)^2/(qx)}, \\
-ix^{3/2}q^{-1/2}\sum_n ne^{-\pi n^2 x/q}\sum_b \bar\chi(b)e^{2\pi inb/q} &= \sum_{n,b}\bar\chi(b)(qn + b)e^{-\pi(qn+b)^2/(qx)}.
\end{aligned}$$

Use the lemma above ($\tau(\bar\chi, n) = \chi(n)\tau(\bar\chi)$) to finally obtain

$$\begin{aligned}
(x/q)^{1/2}\tau(\bar\chi)\psi_0(\chi, x) &= \psi_0(\bar\chi, 1/x), \\
-ix^{3/2}q^{-1/2}\tau(\bar\chi)\psi_1(\chi, x) &= \psi_1(\bar\chi, 1/x).
\end{aligned}$$

□

Theorem (Functional equation for $L(s,\chi)$)**.** *Given a primitive character χ of modulus q, define*

$$\xi(s,\chi) = (\pi/q)^{-\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)$$

where $a = 0, 1$ if $\chi(-1) = 1, -1$. Then ξ satisfies the functional equation

$$\xi(1-s,\bar{\chi}) = \frac{i^a q^{1/2}}{\tau(\chi)}\xi(s,\chi).$$

Proof. Suppose $\chi(-1) = 1$. We proceed as in the construction of the functional equation for ζ starting with

$$\Gamma(s/2) = \int_0^\infty e^{-x}x^{s/2-1}dx,$$

substituting $x = n^2\pi t/q$

$$(\pi/q)^{-s/2}n^{-s}\Gamma(s/2) = \int_0^\infty e^{-n^2\pi t/q}t^{s/2-1}dt,$$

multiplying by $\chi(n)$ and summing over n to get

$$\xi(s,\chi) = (\pi/q)^{-s/2}\Gamma(s/2)L(s,\chi) = \frac{1}{2}\int_0^\infty \psi_0(t,\chi)t^{s/2-1}dt$$

where

$$\psi_0(t,\chi) = \sum_{n\in\mathbb{Z}}\chi(n)e^{-n^2\pi t/q}.$$

Split the integral at $t = 1$ and use the functional equation for ψ_0

$$\begin{aligned}\xi(s,\chi) &= \frac{1}{2}\int_1^\infty \psi_0(t,\chi)t^{s/2-1}dt + \frac{1}{2}\int_1^\infty \psi_0(1/t,\chi)t^{-s/2-1}dt\\ &= \frac{1}{2}\int_1^\infty \psi_0(t,\chi)t^{s/2-1}dt + \frac{\tau(\chi)}{2q^{1/2}}\int_1^\infty \psi_0(t,\bar{\chi})t^{-s/2-1/2}dt\\ &= \frac{\tau(\chi)}{q^{1/2}}\xi(1-s,\bar{\chi})\end{aligned}$$

using the fact that $\tau(\chi)\tau(\bar{\chi}) = \tau(\chi)\overline{\tau(\chi)} = q$ (since χ is even).

Now assume $\chi(-1) = -1$. We start with $\Gamma((s+1)/2)$

$$\Gamma((s+1)/2) = \int_0^\infty e^{-x}x^{(s-1)/2}dx,$$

substituting $x = n^2\pi t/q$

$$(\pi/q)^{-(s+1)/2}n^{-s}\Gamma((s+1)/2) = \int_0^\infty ne^{-n^2\pi t/q}t^{(s-1)/2}dt,$$

multiplying by $\chi(n)$ and summing over n to get

$$\xi(s,\chi) = (\pi/q)^{-(s+1)/2}\Gamma((s+1)/2)L(s,\chi) = \frac{1}{2}\int_0^\infty \psi_1(t,\chi)t^{(s-1)/2}dt$$

where

$$\psi_1(t,\chi) = \sum_{n\in\mathbb{Z}} n\chi(n)e^{-n^2\pi t/q}.$$

Split the integral at $t=1$ and use the functional equation for ψ_1

$$\begin{aligned}\xi(s,\chi) &= \frac{1}{2}\int_1^\infty \psi_0(t,\chi)t^{(s-1)/2}dt + \frac{1}{2}\int_1^\infty \psi_0(1/t,\chi)t^{-(s+3)/2}dt \\ &= \frac{1}{2}\int_1^\infty \psi_0(t,\chi)t^{(s-1)/2}dt + \frac{iq^{1/2}}{2\tau(\bar{\chi})}\int_1^\infty \psi_0(t,\bar{\chi})t^{-s/2}dt \\ &= \xi(1-s,\bar{\chi})\end{aligned}$$

using the fact that $\tau(\chi)\tau(\bar{\chi}) = -\tau(\chi)\overline{\tau(\chi)} = -q$ (since χ is odd). $\square$

Here is another proof of the functional equation, along the lines of the second proof for ζ given above (taken from Brendt).

Theorem. *For χ modulo q primitive, we have*

$$L(1-s,\chi) = q^{s-1}(2\pi)^{-s}\tau(\chi)\Gamma(s)L(s,\bar{\chi})(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}).$$

Proof. Start with Γ, make a change of variable, multiply by $\chi(n)$ and sum over n

$$\begin{aligned}\Gamma(s) &= \int_0^\infty e^{-x}x^{s-1}dx = n^s\int_0^\infty e^{-nx}x^{s-1}dx, \\ L(s,\chi)\Gamma(s) &= \int_0^\infty\left(\sum_n \chi(n)e^{-nx}\right)x^{s-1}dx = \int_0^\infty\left(\sum_a \chi(a)\sum_n e^{-(nq+a)x}\right)x^{s-1}dx \\ &= \int_0^\infty x^{s-1}\sum_n e^{-nqx}\sum_a \chi(a)e^{-ax}dx = \int_0^\infty \tau\left(\chi,\frac{iqx}{2\pi}\right)\frac{x^{s-1}}{1-e^{qx}}dx.\end{aligned}$$

We will calculate the integral using residues. Consider the function

$$F(z) = \frac{\pi e^{-\pi iz}\tau(\bar{\chi},z)}{z^s\sin(\pi z)\tau(\bar{\chi})}$$

and the positively oriented contour C_m consisting of two right semicirlces and the segments connecting them

$\Gamma_m = \{(m+1/2)e^{i\theta} : -\pi/2 \le \theta \le \pi/2\}$, $\Gamma_\epsilon = \{\epsilon e^{i\theta} : -\pi/2 \le \theta \le \pi/2\}$, $\{it : \pm\epsilon \le t \le \pm(m+1)\}$.

F is meromorphic on the interior of C_m with simple poles at the zeros of the sine factor, $z = 1,\dots,m$, with residues

$$\lim_{z\to n}(z-n)F(z) = \frac{e^{-n\pi i}}{n^s}\frac{\tau(\bar{\chi},n)}{\tau(\bar{\chi})}\lim_{z\to n}\frac{\pi(z-n)}{\sin(\pi z)} = (-1)^n\frac{\chi(n)}{n^s}\lim_{z\to n}\frac{\pi(z-n)}{(-1)^n\sin(\pi(z-n))} = \frac{\chi(n)}{n^s}.$$

Hence

$$\frac{1}{2\pi i}\int_{C_m} F(z)dz = \sum_{n=1}^{m}\frac{\chi(n)}{n^s}.$$

For $s > 1$ the integral on Γ_m goes to zero since

$$\left|\frac{\pi e^{-\pi iz}\tau(\bar\chi, z)}{\tau(\bar\chi)\sin(\pi z)}\right| = \left|\frac{2\pi i\tau(\bar\chi, z)}{\tau(\bar\chi)(e^{2\pi iz}-1)}\right| \le 2\pi\sqrt{q}\frac{e^{-2\pi\Im(z)/q}}{|e^{2\pi iz}-1|} \le M$$

is bounded. Letting $m \to \infty$ gives

$$L(s,\chi) = \int_{i\epsilon}^{i\infty}\frac{\tau(\bar\chi, z)dz}{\tau(\bar\chi)z^s(1-e^{2\pi iz})} + \int_{-i\epsilon}^{-i\infty}\frac{e^{-2\pi iz}\tau(\bar\chi, z)dz}{\tau(\bar\chi)z^s(1-e^{-2\pi iz})} + \int_{\Gamma_\epsilon}F(z)dz,$$

and the two infinite integrals converge uniformly on compacta. For $s < 0$, $F(z) \to 0$ as $z \to 0$ since $\tau(\bar\chi, z) \to 0$ and $\sin(\pi z)$ has a simple zero. Hence the integral over Γ_ϵ goes to zero and we get, letting $\epsilon \to 0$ and $z \mapsto iy, -iy$ in the first line, $y \mapsto qy/2\pi$ in the second line,

$$\begin{aligned} L(s,\chi) &= \int_0^\infty \frac{i\tau(\bar\chi, iy)dy}{\tau(\bar\chi)(e^{\pi i/2}y)^s(1-e^{-2\pi y})} + \int_0^\infty \frac{-ie^{-2\pi y}\tau(\bar\chi, -iy)dy}{\tau(\bar\chi)(e^{\pi i/2}y)^s(1-e^{-2\pi y})} \\ &= ie^{-\pi is/2}(q/2\pi)^{1-s}\int_0^\infty \frac{\tau(\bar\chi, iqy/2\pi)dy}{\tau(\bar\chi)y^s(1-e^{-qy})} \\ &\quad - ie^{\pi is/2}(q/2\pi)^{1-s}\int_0^\infty \frac{e^{-qy}\tau(\bar\chi, -iqy/2\pi)dy}{\tau(\bar\chi)y^s(1-e^{-qy})}. \end{aligned}$$

Note that under $j \mapsto q - j$ in the Gauss sum we have

$$\tau(\chi, z) = \sum_j \chi(j)e^{2\pi ijz/q} = \chi(-1)e^{2\pi iz}\sum_j \chi(j)e^{-2\pi ijz/q} = \chi(-1)e^{2\pi iz}\tau(\chi, -z).$$

Using this above we get

$$\begin{aligned} L(s,\chi) &= \frac{i}{\tau(\bar\chi)}(q/2\pi)^{1-s}(e^{-\pi is/2} - \chi(-1)e^{\pi is/2})\int_0^\infty \frac{\tau(\bar\chi, iqy/2\pi)dy}{y^s(1-e^{-qy})} \\ &= \frac{i}{\tau(\bar\chi)}(q/2\pi)^{1-s}(e^{-\pi is/2} - \chi(-1)e^{\pi is/2})\Gamma(1-s)L(1-s,\bar\chi) \end{aligned}$$

upon inspection of our earlier integral representation for $\Gamma(s)L(s,\chi)$. Upon $s \to 1-s$ and using $\tau(\chi)\tau(\bar\chi) = \chi(-1)q$, we get the stated functional equation. □

4 Product Formula for $\xi(s), \xi(s,\chi)$

We would like to establish the product representation

$$\xi(s) = e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho},\ e^A = \xi(0) = \frac{1}{2},\ B = \frac{\xi'(0)}{\xi(0)} = -\frac{\gamma}{2} - 1 + \frac{1}{2}\log 4\pi$$

where the product is over all the zeros of ξ, i.e. the non-trivial zeros of the Riemann zeta function, and a similar product

$$\xi(s,\chi) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

the product over all zeros of $\xi(s,\chi)$, the non-trivial zeros of $L(s,\chi)$.

We have the following theorems of complex analysis.

Theorem (Weierstrass Factorization). *For any entire function with non-zero zeros a_n (repeated with multiplicity) and a zero of order m at zero there exists an entire function g and a sequence of integers p_n such that*

$$f(z) = e^{g(z)} z^m \prod_n E_{p_n}(z/a_n)$$

where

$$E_0(z) = 1 - z, \ \ E_p(z) = (1-z)e^{1+z+z^2/2+\cdots+z^p/p}$$

Conversely, if $|a_n| \to \infty$ is a sequence of non-zero complex numbers and p_n are integers such that

$$\sum_n \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \text{ for all } r > 0,$$

then

$$\prod_n E_{p_n}(z/a_n)$$

is entire with zeros only at a_n (with prescribed multiplicity).

For example

$$\sin(\pi z) = \pi z \prod_{n\in\mathbb{Z}\setminus\{0\}} (1 - z/n)e^{z/n} = \pi z \prod_{n=1}^{\infty}(1 - z^2/n^2),$$

and

$$1/\Gamma(z) = e^{\gamma z} z \prod_{n=1}^{\infty}(1 + z/n)e^{-z/n}$$

We say that an entire function f is of *order* $\lambda < \infty$ if λ is greatest lower bound among λ such that

$$|f(z)| = O\left(e^{|z|^\lambda}\right) \text{ as } |z| \to \infty.$$

Assume that an entire function f has a Weierstrass factorization with $p_n = p$ constant (i.e. there is an integer p such that $\sum_n |a_n|^{-(p+1)} < \infty$) and $g(z)$ a polynomial of degree q. Taking p minimal makes p and $g + 2\pi i\mathbb{Z}$ unique. The *genus* of f is the maximum of p and q. We have the following proposition.

Proposition. *If f is entire of genus μ, then for all $\alpha > 0$ and for $|z|$ large enough, we have*

$$|f(z)| \leq e^{\alpha |z|^{\mu+1}},$$

i.e. an entire function f of genus μ has order less or equal $\mu + 1$.

The proof of the proposition depends on the following.

Theorem (Jensen's Formula)**.** *If f is entire with zeros z_i inside $|z| < R$, $f(0) \neq 0$, and $f(z) \neq 0$ on $|z| = R$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta = \log |f(0)| + \log \frac{R^n}{|z_1| \cdot \cdots \cdot |z_n|} = \int_0^R \frac{n(r)}{r} dr$$

where $n(r)$ is the number of zeros of f of absolute value less than r.

One consequence of Jensen's formula is that if f is of order λ and $\alpha > \lambda$, then $\sum_i |z_i|^{-\alpha} < \infty$. To see this, note that $n(R) = O(R^\alpha)$ since

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)| \leq R^\alpha - \log |f(0)|$$

and

$$\int_R^{2R} \frac{n(r)}{r} dr \geq n(R) \log 2$$

so that

$$n(R) \log 2 \leq \int_0^{2R} \frac{n(r)}{r} dr \leq (2R)^\alpha - \log |f(0)| = O(R^\alpha).$$

From this we see that for $\beta > \alpha > \lambda$

$$\sum_i |z_i|^{-\beta} = \int_0^\infty r^{-\beta} dn(r) = \beta \int_0^\infty r^{-\beta-1} n(r) dr < \infty.$$

The converse of the above proposition holds, showing that entire functions of finite order have nice factorizations.

Theorem (Hadamard Factorization)**.** *An entire function f of order λ has finite genus $\mu \leq \lambda$.*

We apply the above to the entire functions $\xi(s), \xi(s, \chi)$. We will need the following to estimate their rates of growth.

Theorem (Stirling's Formula)**.** *For $z \in \mathbb{C} \backslash (-\infty, 0]$ we have*

$$\Gamma(z) = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z e^{\mu(z)}$$

where

$$\mu(z) = -\int_0^\infty \frac{\{t\} - 1/2}{z+t} dt = \int_0^\infty \frac{1}{2} \frac{\{t\} - \{t\}^2}{(z+t)^2} dt$$

with $\{t\} = t - \lfloor t \rfloor$ the fractional part of t, and bounds on μ given by

$$|\mu(z)| \leq \frac{1}{8} \frac{1}{\cos^2(\theta/2)} \frac{1}{|z|}, \;\; z = |z| e^{i\theta}.$$

Lemma. *The entire functions* $\xi(s), \xi(s,\chi)$, *have order* 1.

Proof. Since $\xi(s) = \xi(1-s)$ we consider $\sigma \geq 1/2$, where we have

$$\left|\frac{s(s-1)}{2}\pi^{-s/2}\right| \leq e^{C|s|},$$
$$|\Gamma(s/2)| \leq e^{C|s|\log|s|},$$
$$|\zeta(s)| = \left|\frac{s}{s-1} + s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx\right| \leq C|s|,$$

using Stirling's formula and an integral representation of $\zeta(s)$ applicable for $\sigma > 0$. Hence $\xi(s)$ has order at most 1, actually equal to 1 since for real $s \to \infty$ we have $\zeta(s) \to 1$ and $\log\Gamma(s) \sim s\log s$.

Similarly if χ is a primitive character modulo q, then for $\xi(s,\chi) = (q/\pi)^{(s+a)/2)}\Gamma((s+a)/2)L(s,\chi)$ with functional equation $\xi(1-s,\bar{\chi}) = \frac{i^a\sqrt{q}}{\tau(\chi)}\xi(s,\chi)$, and for $\sigma \geq 1/2$ we have

$$\left|(q/\pi)^{(s+a)/2)}\right| \leq e^{C|s|},$$
$$|\Gamma((s+a)/2)| \leq e^{C|s|\log|s|},$$
$$|L(s,\chi)| = \left|s\int_1^\infty \frac{\sum_{n\leq x}\chi(n)}{x^{s+1}}dx\right| \leq C|s|,$$

so that $|\xi(s,\chi)| \leq q^{(\sigma+1)/2}e^{C|s|\log|s|}$ (for $\sigma > 1/2$, similar results for $\sigma < 1/2$ by the functional equation). Hence $\xi(s,\chi)$ is of order 1 as well. □

From the general theory above, we have the desired product formulae for $\xi(s), \xi(s,\chi)$.

Although the constants A, B are not of much importance, we calculate them for ξ anyway. For the constant A we have

$$e^A = \xi(0) = \xi(1) = \frac{1}{2\sqrt{\pi}}\Gamma(1/2)\lim_{s\to 1}(s-1)\zeta(s) = 1/2.$$

For the constant B we have

$$B = \frac{\xi'(0)}{\xi(0)} = -\frac{\xi'(1)}{\xi(1)}$$

so we consider

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_\rho \frac{1}{s-\rho} + \frac{1}{\rho} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} - \log\sqrt{\pi} + \frac{1}{2}\frac{\Gamma'(1+s/2)}{\Gamma(1+s/2)},$$

and

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{s+n}\right).$$

Hence

$$\frac{1}{2}\frac{\Gamma'(3/2)}{\Gamma(3/2)} = -\frac{\gamma}{2} - \frac{1}{3} + \sum_{n\geq 1}\left(\frac{1}{2n} - \frac{1}{3+2n}\right) = -\frac{\gamma}{2} - 1 + \sum_{n\geq 2}\frac{(-1)^n}{n} = -\frac{\gamma}{2} + 1 - \log 2$$

and

$$B = -\frac{\gamma}{2} - 1 + \log\sqrt{4\pi} - \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right).$$

With $I(s) = \int_1^\infty \{x\}/x^{s+1}dx$ we have

$$\zeta(s) = \frac{s}{s-1}(1-(s-1)I(s))$$

so that

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \frac{1}{s} - \frac{(s-1)I'(s)+I(s)}{1-(s-1)I(s)}$$

and

$$\lim_{s\to 1}\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = 1 - I(1).$$

Finally note that

$$\begin{aligned} I(1) &= \int_1^\infty \frac{x-\lfloor x\rfloor}{x^2}dx = \lim_{N\to\infty}\int_1^N \frac{dx}{x} - \sum_{n=1}^{N-1} n\int_n^{n+1}\frac{1}{x^2} \\ &= \lim_{N\to\infty}\log N - \sum_{n=1}^{N-1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 + \lim_{N\to\infty}\log N - \sum_{n=1}^{N}\frac{1}{n} \\ &= 1-\gamma, \end{aligned}$$

so that

$$B = -\frac{\gamma}{2} - 1 + \log\sqrt{4\pi}.$$

Another expression for B in terms of the zeros of ξ is

$$B = -\frac{1}{2}\sum_\rho \frac{1}{\rho(1-\rho)}$$

which can be seen from the equation (using $s \leftrightarrow 1-s$ and $\rho \leftrightarrow 1-\rho$)

$$B + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = -B - \sum_\rho\left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right)$$

from ξ'/ξ or from the product itself (say at $s=0$)

$$e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho} = e^{A+B(1-s)}\prod_\rho\left(1-\frac{1-s}{1-\rho}\right)e^{(1-s)/(1-\rho)}.$$

Using this we can write the product formula (similar to that of $\sin(\pi z)$ combining the roots $\pm n$) as

$$\xi(s) = \xi(0)\prod_{\Im\rho>0}\left(1-\frac{s(1-s)}{\rho(1-\rho)}\right).$$

5 A Zero-Free Region and the Density of Zeros for $\zeta(s)$

We would like to extend the zero-free region of the zeta function to an open set containing $\sigma \geq 1$. Specifically we have the following.

Theorem. *There is a $c > 0$ such that $\Re\rho < 1 - c/\log(|t| + 2)$ for any zero ρ of $\zeta(s)$.*

Proof. Once again we make use of

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$$

applied to

$$-\Re\frac{\zeta'(s)}{\zeta(s)} = \sum_n \frac{\Lambda(n)}{n^\sigma}\cos(t\log n).$$

Considering the logarithmic derivative via the product formula for ξ we have

$$\frac{\zeta'(s)}{\zeta(s)} = B + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'(1+s/2)}{\Gamma(1+s/2)} + \frac{1}{2}\log\pi.$$

We obtain the following estimates (say for $1 \leq \sigma \leq 2, |t| \geq 2$, using A to represent a positive constant, not the same at each instance)

$$\begin{aligned} -\frac{\zeta(\sigma)}{\zeta(\sigma)} &< \frac{1}{\sigma - 1} + A, \\ -\Re\frac{\zeta(\sigma+it)}{\zeta(\sigma+it)} &< A\log|t| - \frac{1}{\sigma-\beta}, \\ -\Re\frac{\zeta(\sigma+2it)}{\zeta(\sigma+2it)} &< A\log|t|, \end{aligned}$$

where $\rho = \beta + i\gamma$ is a zero of zeta with $\gamma = t$, using the facts that $\Gamma'(s)/\Gamma(s) \leq A\log|t|$ (PROOF???) and that the sum over the roots is positive

$$\Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = \frac{\sigma-\beta}{|s-\rho|^2} + \frac{\beta}{|\rho|^2}.$$

Hence we obtain

$$\begin{aligned} 3\left(-\frac{\zeta(\sigma)}{\zeta(\sigma)}\right) + 4\left(-\Re\frac{\zeta(\sigma+it)}{\zeta(\sigma+it)}\right) + \left(-\Re\frac{\zeta(\sigma+2it)}{\zeta(\sigma+2it)}\right) &\geq 0, \\ 3\left(\frac{1}{\sigma-1} + A\right) + 4\left(A\log|t| - \frac{1}{\sigma-\beta}\right) + (A\log|t|) &\geq 0, \\ \frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + A\log|t| &\geq 0. \end{aligned}$$

Let $\sigma = 1 + \delta/\log|t|$ for some positive δ. Then

$$\beta \le 1 - \frac{\delta - A\delta^2}{\log|t|}$$

and choosing δ so that $\delta - A\delta^2 > 0$ we have $\beta \le 1 - c/\log|t|$ for some $c > 0$. Combining this with the fact that ζ has no zeros in the region $1 \le \sigma \le 2, |t| < 2$ gives the result. □

We also have the following estimate stated by Riemann about the density of zeros in the critical strip $0 < \sigma < 1$.

Theorem. *Let $N(T)$ be the number of zeros of $\zeta(s)$ in the region $0 < \sigma < 1, 0 < t < T$. Then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

as $T \to \infty$.

Proof. Let R be the rectangle with vertices $\{-1, 2, 2+iT, -1+iT\}$. Then if T doesn't coincide with the ordinate of a zero, we have

$$2\pi N(T) = \Delta_R \arg\xi(\text{change in the argument}).$$

Since $\xi(s) = \xi(1-s) = \overline{\xi(1-\bar{s})}$ and xi is real on the real axis, we have $\pi N(T) = \Delta_L \arg\xi$ where L is the segment running between $2, 2+iT, 1/2+iT$. With

$$\xi(s) = \pi^{-s/2}(s-1)\Gamma(1+s/2)\zeta(s)$$

and

$$\log\Gamma(s) = (s-1/2)\log s - s + \log(\sqrt{2\pi}) + O(1/s)$$

we have

$$\begin{aligned}
\Delta_L \arg(s-1) &= \arg(iT - 1/2) = \frac{\pi}{2} + \arctan\left(\frac{1}{2T}\right) = \frac{\pi}{2} + O(1/T),\\
\Delta_L \arg\pi^{-s/2} &= \arg\left(e^{-(1/2+iT)\log(\pi)/2}\right) = -\frac{T}{2}\log(\pi),\\
\Delta_L \arg\Gamma(1+s/2) &= \Im\log\Gamma\left(\frac{5}{4} + \frac{T}{2}i\right)\\
&= \Im\left[\left(\frac{3}{4} + \frac{T}{2}i\right)\log\left(\frac{5}{4} + \frac{T}{2}i\right) - \left(\frac{5}{4} + \frac{T}{2}i\right) + \log(\sqrt{2\pi}) + O(1/T)\right]\\
&= \frac{T}{2}\log\left(\frac{T}{2}\right) - \frac{T}{2} + \frac{3\pi}{8} + O(1/T),
\end{aligned}$$

(using

$$\log\left(\frac{5}{4} + \frac{T}{2}i\right) = \log\left(\frac{T}{2}\right) + O(1/T^2) + \left(\frac{\pi}{2} + O(1/T)\right)i.$$

in the last step). So we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \Delta_L \arg\zeta + O(1/T).$$

We now show that $\Delta_L \arg\zeta = \arg\zeta(1/2+iT) = O(\log T)$. □

6 A Zero-Free Region and the Density of Zeros for $L(s, \chi)$

7 Explicit Formula Relating the Primes to Zeros of $\zeta(s)$

We first want to establish a formula explicitly relating the primes to the zeros of the Riemann zeta function, namely

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{1}{2}\log(1 - x^{-2}) - \frac{\zeta'(0)}{\zeta(0)}$$

where ψ_0 is the Chebychev function $\psi(x)$, but taking its average value at discontinuities

$$\psi_0(x) = \begin{cases} \psi(x) = \sum_{n \leq x} \Lambda(n) & x \text{ not a prime power} \\ \psi(x) - \frac{1}{2}\Lambda(x) & x \text{ a prime power} \end{cases}$$

(here $\Lambda(n) = \log p$ if $n = p^k, k \geq 1$ is a prime power, zero otherwise, is the von Mangoldt function).

The sum over the non-trivial zeros of the zeta function is conditionally convergent, so we pair $\rho, \bar{\rho}$. Also note that the log term is the sum over the trivial zeros of zeta,

$$-\sum_n \frac{x^{-2n}}{n} = \log(1 - x^{-2})$$

and that $\zeta'(0)/\zeta(0) = \log(2\pi)$: using ξ'/ξ, the derivation of the constant B and the fact that $\Gamma'(1)/\Gamma(1) = -\gamma$ we have

$$\begin{aligned} \frac{\zeta'(0)}{\zeta(0)} &= B + 1 + \log\sqrt{\pi} - \frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)} \\ &= \left(-\frac{\gamma}{2} - 1 + \log\sqrt{4\pi}\right) + 1 + \log\sqrt{\pi} + \frac{\gamma}{2} \\ &= \log(2\pi). \end{aligned}$$

We obtain this forumla by evaluating the integral

$$\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s)} x^s \frac{ds}{s}$$

in two differnent ways. From the Euler product, the logarithmic derivative of ζ is intimately related to the von Mangoldt function

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\log\prod_p \left(1 - p^{-s}\right)^{-1} = -\sum_{p,n>0} p^{-ns}\log p = -\sum_n \frac{\Lambda(n)}{n^s}.$$

From the product expression for ξ, we can express ζ'/ζ in terms of the zeros of zeta,

$$\begin{aligned}\frac{d}{ds}\log\xi(s) &= B + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)\\ &= \frac{d}{ds}\log\big((s-1)\pi^{-s/2}\Gamma(1+s/2)\zeta(s)\big)\\ &= \frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{2}\frac{\Gamma'(1+s/2)}{\Gamma(1+s/2)}-\frac{1}{2}\log\pi,\end{aligned}$$

so that

$$\frac{\zeta'(s)}{\zeta(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'(1+s/2)}{\Gamma(1+s/2)} + \frac{1}{2}\log\pi.$$

We need the following lemma.

Lemma. *Let $c, y > 0$. Then*

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s\frac{ds}{s} = \begin{cases} 0 & 0<y<1\\ 1/2 & y=1\\ 1 & y>1\end{cases}.$$

More specifically, if

$$\delta(y) = \begin{cases} 0 & 0<y<1\\ 1/2 & y=1\\ 1 & y>1\end{cases},\quad I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s\frac{ds}{s},$$

then

$$|I(y,T)-\delta(y)| \le \begin{cases} y^c\min\{1,(T|\log y|)^{-1}\} & y\neq 1\\ cT^{-1} & y=1\end{cases}.$$

Proof. For one of the inequalities, we consider the integral $(2\pi i)^{-1}\int_R y^s ds/s$ around a large rectangle R with corners $\{c\pm iT, C\pm iT\}$ and let $C\to\pm\infty$ depending on whether $0<y<1$ or $y>1$, the integral along the vertical edge at infinity being zero in each case respectively.

For $0<y<1$, we have $(2\pi i)^{-1}\int_R y^s ds/s = 0$ so that

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{\infty-iT}\frac{y^s}{s}ds - \frac{1}{2\pi i}\int_{c+iT}^{\infty+iT}\frac{y^s}{s}ds$$

and

$$|I(y,T)-\delta(y)| < \frac{1}{T}\int_c^\infty y^\sigma d\sigma = \frac{-\pi y^c}{\log y}.$$

For $y>1$, we have $(2\pi i)^{-1}\int_R y^s ds/s = 1$ so that

$$I(y,T) = 1 - \frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT}\frac{y^s}{s}ds + \frac{1}{2\pi i}\int_{-\infty+iT}^{c+iT}\frac{y^s}{s}ds$$

and

$$|I(y,T)-\delta(y)| < \frac{1}{T}\int_{-\infty}^{c} y^\sigma\, d\sigma = \frac{\pi y^c}{\log y}.$$

For the other inequality, we use a circular contour of radius $R=(c^2+T^2)^{1/2}$ centered at the origin where

$$\left|\frac{y^s}{s}\right| = \frac{y^{R\cos\theta}}{R} \le \frac{y^c}{R} \text{ for either of } 0<y<1,\ 1<y$$

to see that

$$|I(y,T)-\delta(y)| \le 2\pi R\frac{1}{2\pi}\frac{y^c}{R} = y^c.$$

Finally, for the case $y=1$ we have

$$\begin{aligned}
I(1,T) &= \frac{1}{2\pi i}\int_{-T}^{T}\frac{d(c+it)}{c+it} = \frac{1}{2\pi}\int_0^T\left(\frac{1}{c+it}-\frac{1}{c-it}\right)dt = \frac{1}{\pi}\int_0^T\frac{c}{c^2+t^2}dt\\
&= \frac{1}{\pi}\int_0^{T/c}\frac{du}{1+u^2} = \frac{1}{2} - \int_{T/c}^{\infty}\frac{du}{1+u^2},\\
\left|I(1,T)-\frac{1}{2}\right| &= \frac{1}{\pi}\int_{T/c}^{\infty}\frac{du}{1+u^2} \le \int_{T/c}^{\infty}\frac{du}{u^2} = \frac{c}{T}.
\end{aligned}$$

□

Using the lemma we have

$$\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta'(s)}{\zeta(s)}x^s\frac{ds}{s} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_n\frac{\Lambda(n)}{n^s}x^s\frac{ds}{s} = \psi_0(x),$$

while the evaluation of the integral using the other expression for ζ'/ζ gives, for $x>1$,

$$\begin{aligned}
&\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta'(s)}{\zeta(s)}x^s\frac{ds}{s}\\
=&\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(B+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'(1+s/2)}{\Gamma(1+s/2)}+\frac{1}{2}\log\pi\right)x^s\frac{ds}{s}\\
=&\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\log(2\pi)-\frac{s}{s-1}+\sum_\rho\frac{s}{\rho(s-\rho)}-\sum_{n\ge1}\frac{s}{2n(s+2n)}\right)x^s\frac{ds}{s}\\
=&-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s ds}{s-1}-\sum_\rho\frac{1}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s ds}{s-\rho}+\sum_{n\ge1}\frac{1}{2n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s ds}{s+2n}\\
=&-\frac{\zeta'(0)}{\zeta(0)}+x\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}ds}{s-1}-\sum_\rho\frac{x^\rho}{\rho}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-\rho}ds}{s-\rho}+\sum_{n\ge1}\frac{x^{-2n}}{2n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+2n}ds}{s+2n}\\
=&-\frac{\zeta'(0)}{\zeta(0)}+x-\sum_\rho\frac{x^\rho}{\rho}+\sum_{n\ge1}\frac{x^{-2n}}{2n},
\end{aligned}$$

(modulo a whole bunch of convergence).

8 Chebyshev Estimates and the Prime Number Theorem

We start with some elementary estimates, bounding $\pi(x)$, the number of primes less or equal x.

Theorem (Chebyshev Estimates)**.** *There are constants* $0 < c_1 \leq 1 \leq c_2$ *such that*

$$\frac{c_1 x}{\log x} \leq \pi(x) \leq \frac{c_2 x}{\log x}.$$

Proof. For an upper bound, we start with

$$\prod_{n<p<2n} p < \binom{2n}{n} < 2^{2n}$$

so that, for $\vartheta(x) = \sum_{p \leq x} \log p$ we have

$$\vartheta(2n) - \vartheta(n) = \sum_{n<p<2n} \log p \leq 2n \log 2.$$

Summing over $n = 2^k, 0 \leq k \leq 2^{m-1}$ gives

$$\vartheta(2^m) = \sum_{k=0}^{m-1} \vartheta(2 \cdot 2^k) - \vartheta(2^k) \leq \sum_{k=0}^{m-1} 2^{k+1} \log 2 \leq 2^{m+1} \log 2.$$

For $2^{m-1} < x \leq 2^m$ we have

$$\vartheta(x) \leq \vartheta(2^m) \leq 2^{m+1} \log 2 = (4 \log 2) 2^{m-1} \leq (4 \log 2) x.$$

[A similar/equivalent estimate for ψ is obtained by considering

$$S(x) = \sum_{n \leq x} \log n = \sum_{n \leq x} \psi(x/n) = x \log x - x + O(\log x)$$

since

$$\sum_{n \leq x} \log n = \int_1^x \left(\log t + \frac{W(t)}{t} \right) dt + \frac{1}{2} \log x \leq x \log x - x + \log x$$

by Euler-Maclurin summation. We have

$$\begin{aligned} S(x) - 2S(x/2) &= -\sum_n (-1)^n \psi(x/n) \\ &= x \log x - x + O(\log x) - 2 \left(\frac{x}{2} \log(x/2) - \frac{x}{2} + O(\log(x/2)) \right) \\ &= x \log 2 + O(\log x). \end{aligned}$$

Hence $\psi(x) > x\log 2 + O(\log x)$ and $\psi(x) - \psi(x/2) < x\log 2 + O(\log x)$. If r is maximal such that $x/2^r \geq 2$ then

$$\psi(x) - \psi(x/2^r) = \sum_{i=0}^{r-1} \psi(x/2^i) - \psi(x/2^{i+1}) \leq x2\log 2 + rO(\log x) = x\log 4 + O((\log x)^2)$$

since $r = O(\log x)$. Hence

$$x\log 2 + O(\log x) < \psi(x) < x\log 4 + O((\log x)^2)$$

showing $\psi(x) \asymp x$.]

We relate this to $\pi(x)$ by noting that for $0 < \alpha < 1$ we have

$$[\pi(x) - \pi(x^\alpha)]\log x^\alpha \leq \vartheta(x) - \vartheta(x^\alpha) \leq \vartheta(x) < (4\log 2)x$$

so that

$$\pi(x) \leq \frac{(4\log 2)x}{\alpha\log x} + \pi(x^\alpha) \leq \frac{(4\log 2)x}{\alpha\log x} + x^\alpha = \frac{x}{\log x}\left(\frac{4\log 2}{\alpha} + x^{\alpha-1}\log x\right) \leq \frac{6x}{\log x}$$

(for instance, choosing α wisely).

For a lower bound, we start with

$$2^n < \binom{n+1}{1} \cdot \binom{n+2}{2} \cdot \ldots \binom{n+n}{n} = \binom{2n}{n}.$$

We need the fact (easy to prove) that the largest power of p dividing $n!$, $\mathrm{ord}_p(n!)$, is given by $\sum_{k\geq 1}\lfloor n/p^k\rfloor$. This gives

$$\mathrm{ord}_p\binom{2n}{n} = \sum_{k\geq 1}\left(\left\lfloor\frac{2n}{p^k}\right\rfloor - 2\left\lfloor\frac{n}{p^k}\right\rfloor\right) \leq \frac{\log(2n)}{\log p}$$

since each term in the sum is 0 or 1 and for $p^k > 2n$ we have $\lfloor 2n/p^k\rfloor, \lfloor n/p^k\rfloor = 0$. Hence

$$2^n < \binom{2n}{n} \leq \prod_{p\leq 2n} p^{\log 2n/\log p} = (2n)^{\pi(2n)},$$

and taking logarithms gives

$$\pi(2n) \geq \frac{\log 2}{2}\frac{2n}{\log(2n)}.$$

For odd integers we have

$$\pi(2n+1) \geq \pi(2n) \geq \frac{\log 2}{2}\frac{2n}{2n+1}\frac{2n+1}{\log(2n+1)}$$

so that

$$\pi(x) \geq \frac{x/6}{\log x}$$

(for instance). □

We can relate the asymptotics of $\pi(x)$ to those of $\psi(x), \vartheta(x)$ as follows.

Proposition. *$\psi(x), \vartheta(x) \sim x$ if and only if $\pi(x) \sim x/\log x$.*

Proof. In one direction we have

$$\vartheta(x) \leq \psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \leq \pi(x) \log x,$$

while in the other we have, for $0 < \delta < 1$,

$$\psi(x) \geq \vartheta(x) \geq \sum_{x^{1-\delta} \leq p \leq x} \log p \geq (1-\delta)[\pi(x) - \pi(x^{1-\delta})] \log x = (1-\delta) \log x [\pi(x) + O(x^{1-\delta})].$$

Hence for all $0 < \delta < 1$ we have

$$(1-\delta)\frac{\pi(x)}{x/\log x} + O(x^{-\delta} \log x) \leq \frac{\psi(x)}{x}, \ \frac{\vartheta(x)}{x} \leq \frac{\pi(x)}{x/\log x}.$$

□

A 'quick' proof of the prime number theorem comes from the fact that $\zeta(s) \neq 0$ on $\sigma \geq 1$ and the following Tauberian theorem.

Theorem (Newman's Tauberian Theorem)**.** *Suppose $f(t)$ is bounded and locally integrable for $t \geq 0$ and that*

$$g(z) = \int_0^\infty e^{-zt} f(t) dt, \ \Re z > 0$$

extends to a holomorphic funtion on $\Re z \geq 0$. Then

$$\int_0^\infty f(t) dt$$

exists.

Proof. Let C be the boundary of

$$\{|z| \leq R\} \cap \{\Re z \geq -\delta\}$$

where R is large positive and $\delta > 0$ small enough so that $g(z)$ is analytic on C. For $g_T(z) = \int_0^T f(t) e^{-zt} dt$ we have

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}.$$

On the right semi-cricle C_+ we have

$$|g(z) - g_T(z)| = \left| \int_T^\infty f(t) e^{-zt} dt \right| \leq M \int_T^\infty |e^{zt}| dt = \frac{M e^{T \Re z}}{\Re z},$$

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| = \frac{2 \Re z}{R^2} e^{T \Re z},$$

$$\left| \frac{1}{2\pi i} \int_{C_+} (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| \leq \frac{M}{R}$$

where $M = \max_{t\geq 0}\{|f(t)|\}$.

On $C_- = C \cap \{\Re z \leq 0\}$ we estimate integrals involving g, g_T separately. For C'_- the left semi-circle of radius R, we have (since g_T entire)

$$\frac{1}{2\pi i}\int_{C_-} g_T(z)e^{zT}\left(1+\frac{z^2}{R^2}\frac{dz}{z}\right) = \frac{1}{2\pi i}\int_{C'_-} g_T(z)e^{zT}\left(1+\frac{z^2}{R^2}\frac{dz}{z}\right)$$

and estimates

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}dt\right| \leq M\int_{-\infty}^T |e^{-zt}|dt = M\frac{e^{-T\Re z}}{-\Re z},$$
$$\left|e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\right| = \frac{-2\Re z}{R^2}e^{T\Re z} \text{ (as before)},$$
$$\left|\frac{1}{2\pi i}\int_{C'_-} g_T(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq \frac{M}{R}.$$

For the integral involving g we have

$$\left|\frac{1}{2\pi i}\int_{C'_-} g(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| \to 0,\ T\to\infty$$

since the only dependence on T in the integrand is e^{zT}, which quickly approaches zero as $T \to \infty$.

Hence

$$\lim_{T\to\infty} |g(0) - g_T(0)| \leq \frac{2M}{R}$$

for arbitrary R, so that $g_T(0) \to g(0)$ as $T \to \infty$ as desired. □

Theorem (Prime Number Theorem)**.** *$\vartheta(x) \sim x$.*

Proof. The function $\Phi(s) := \sum_p \log p/p^s$ extends meromorphically to $\sigma > 1/2$ with a simple pole at 1 with residue 1 and poles at zeros of $\zeta(s)$ since

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_n \frac{\Lambda(n)}{n^s} = \sum_p \frac{\log p}{p^s - 1} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}$$

and the last sum converges for $\sigma > 1/2$. Because ζ has no zeros in $\sigma \geq 1$, $\Phi(s) - (s-1)^{-1}$ is holomorphic on $\sigma \geq 1$. This along with $\vartheta(x) = O(x)$ allows us to apply the Tauberian theorem to conclude that

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2}dx$$

converges as follows. We have

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = \int_1^\infty \frac{d\vartheta(x)}{x^s} = s\int_1^\infty \frac{\vartheta(x)}{x^{s+1}}dx = s\int_0^\infty e^{-st}\vartheta(e^t)dt,$$
$$\int_0^\infty (\vartheta(e^t)e^{-t} - 1)e^{-zt}dt = \frac{\Phi(z+1)}{z+1} - \frac{1}{z} =: g(z),$$

so that with $f(t) := \vartheta(e^t)e^{-t} - 1$ (remember $\vartheta(x) = O(x)$ so that f is bounded) we have the existence of

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2} dx = \int_0^\infty (\vartheta(e^t)e^{-t} - 1)dt = \int_0^\infty f(t)dt.$$

Finally we show that $\vartheta(x) \sim x$. If not, say $\vartheta(x) \geq \lambda x$ for arbitrarily large x and some $\lambda > 1$ then

$$\int_x^{\lambda x} \frac{\vartheta(t) - t}{t^2} dt \geq \int_x^{\lambda x} \frac{\vartheta(x) - t}{t^2} dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt = \int_1^{\lambda} \frac{\lambda - t}{t^2} dt > 0$$

contradicting convergence of the integral for large x and fixed λ.

Similarly, if $\vartheta(x) \leq \lambda x$ for arbitrarily large x and some $\lambda < 1$ then

$$\int_{\lambda x}^{x} \frac{\vartheta(t) - t}{t^2} dt \leq \int_x^{\lambda x} \frac{\vartheta(x) - t}{t^2} dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2} dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2} dt < 0$$

contradicting convergence of the integral as well. □

Another approach is to show $\psi(x) \sim x$ starting with von Mangoldt's explicit formula

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{1}{2}\log(1 - x^{-2}) - \frac{\zeta'(0)}{\zeta(0)}$$

and using the zero-free region near $\sigma = 1$ to control the sum over the non-trivial zeros of zeta.

Theorem (Prime Number Theorem)**.** $\psi(x) \sim x$.

Proof. Let the zeros be denoted by $\rho = \beta + \gamma i$. We know that there is a constant c such that $\beta \leq 1 - c/\log T$ for $|\gamma| \leq T$, so that

$$|x^\rho| = x^\beta \leq x e^{-c\log(x)/\log T}.$$

We also have

$$\sum_{0<\gamma\leq T} \frac{1}{|\rho|} \leq \sum_{0<\gamma\leq T} \frac{1}{\gamma} = \int_0^T \frac{dN(t)}{t} = \frac{N(t)}{T} + \int_0^T \frac{N(t)}{t^2} dt \ll \log T + \int_0^T \frac{t\log t}{t^2} \ll (\log T)^2$$

where $N(t) \ll t\log t$ is the number of zeros in the critical strip with $0 < \gamma \leq t$. Hence

$$\left| \sum_{|\gamma|\leq T} \frac{x^\rho}{\rho} \right| \ll x(\log T)^2 e^{-c\log x/\log T}.$$

Dividing by x and letting $T \to \infty, x \to \infty$ gives $\psi(x) \sim x$. □

9 Prime Number Theorem in Arithmetic Progressions

We know that for $(a, q) = 1$, there are infinitely many primes congruent to a modulo q and in fact we have the following.

Theorem (PNT in Arithmetic Progresions)**.** *We have*

$$\vartheta(x; a, q) \sim x$$

where

$$\vartheta(x; a, q) = \phi(q) \sum_{\substack{p \leq x \\ p \equiv a(q)}} \log p.$$

From this it follows that

$$\pi(x; a, q) \sim \frac{1}{\phi(q)} \frac{x}{\log x},$$

where $\pi(x; a, q) = |\{p \leq x, p \equiv a(q)\}|$.

Proof. We proceed using Newman's Tauberian theorem as above. We have $\vartheta(x; a, q) = O(x)$ since

$$\vartheta(x; a, q) \leq \phi(q)\vartheta(x) = O(x).$$

We also know that $L(s, \chi) \neq 0$ on $\sigma \geq 1$. □

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