Taylor[1] polynomials

The best constant approximation to a function $f(x)$ near $x = a$ is the constant function

$$T_0(x) = f(a)$$

The best linear approximation to f near $x = a$ is the tangent line approximation

$$T_1(x) = f(a) + f'(a)(x-a)$$

which uses the value of f at a and the value of the first derivative of f at a.

The nth degree Taylor polynomial of f centered at $x = a$ is the best nth degree polynomial approximation to f near $x = a$

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \end{aligned}$$

assuming f is n times differentiable at $x = a$. [The notation $f^{(n)}$ means the nth derivative of f, and $f^{(0)}$ means the zeroth derivative, i.e. the function itself.] The Taylor polynomial T_n is determined by the numbers

$$f(a),\ f'(a),\ \ldots,\ f^{(n)}(a).$$

The Taylor polynomial T_n is the unique nth degree polynomial whose derivatives of order $\leq n$ at $x = a$ agree with the derivatives of f at $x = a$:

$$T_n(a) = f(a),\ T_n'(a) = f'(a),\ T''(a) = f''(a), \ldots,\ T_n^{(n)}(a) = f^{(n)}(a).$$

By "best" approximation, we mean

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0.$$

In fact we have

$$f(x) - T_n(x) = \frac{1}{n!}\int_a^x (x-t)^{n+1} f^{(n+1)}(t)dt = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some z between x and a. [The first of these follows from integration by parts, and the second is a version of the mean value theorem. Proofs can be found here.]

Find the fifth degree Taylor polynomials centered at $a = 0$ for the following functions. Try to establish a pattern for the coefficients and express $T_n(x)$ in summation notation.

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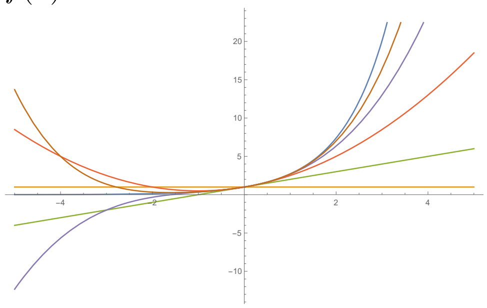
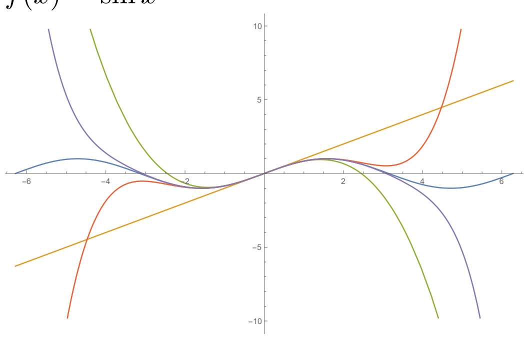
[1] Brook Taylor, (1685-1731)

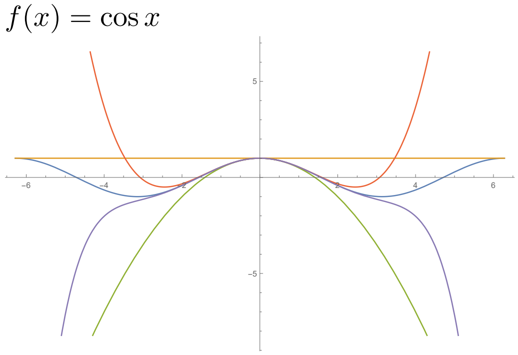
$f(x) = e^x$



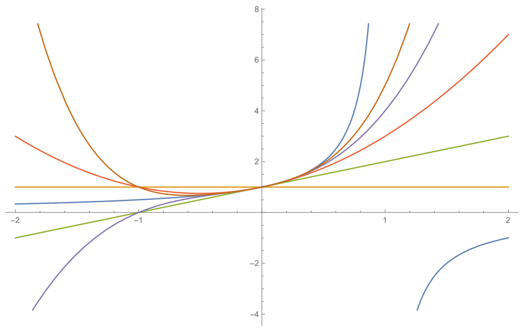
$f(x) = \sin x$



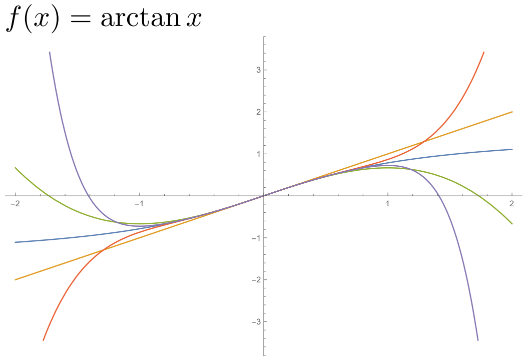
$f(x) = \cos x$



$f(x) = \frac{1}{1-x}$



$f(x) = \arctan x$



$f(x) = -\ln(1 - x)$