Exponential growth and decay

The initial value problem

$$y' = ky,\ y(0) = y_0,$$

describes exponential growth or decay (depending on the sign of the relative growth rate k). This equation is separable, and we have the solution

$$y(x) = y_0 e^{kx}.$$

1. Radium 226 has a half-life of 1590 years. Let $m(t)$ denote the amount of radium after t years in a sample with $m(0) = 100$ g. Answer the following questions:

   - Find $m(t)$.
   - How much of the sample is radium after 1000 years?
   - How much time must pass until there are only 30 g of radium left in the sample?

   We know that $m(t) = 100e^{kt}$ g (using $m(0) = 100$ g), and the relative growth rate k can be found using the half-life:

   $$50 = m(0)/2 = m(1590) = 100e^{1590k},\ k = \frac{-\ln 2}{1590}.$$

   It follows that after 1000 years

   $$m(1000) = 100e^{1000k} = 64.6655\text{g}.$$

   We can also solve for t in the following

   $$m(t) = 30 = 100e^{kt},\ t = \frac{1590\ln(3/10)}{-\ln 2} = 2761.77 \text{ years},$$

   to find how long it takes until only 30 g of radium 226 remain in the sample.

2. Newton's law of cooling states that the rate of change of the temperature $T(t)$ of an object in a constant temperature environment of temperature T_s is proportional the the temperature difference:

$$\frac{dT}{dt} = k(T - T_s),\ k,\ T_s \text{ constants.}$$

- Solve the differential equation above.
- Suppose a can of pop at 72° is put into a refrigerator whose temperature is 44° and that after 30 minutes, the temperature of the can is 61°. Find the temperature of the can $T(t)$ as a function of time.
- What is the temperature of the can after one hour?
- How long must you wait until the temperature of the can is 50°?

Dividing by $(T - 44)$ and integrating gives

$$\int \frac{dT}{T-44} = \int k dt,\ T(t) = 44 + Ce^{kt}.$$

The initial condition $T(0) = 72$ tells us that $C = 72 - 44 = 28$. The information $T(1/2) = 61$ (t in hours) gives k:

$$61 = T(1/2) = 44 + 28e^{k/2},\ k = 2\ln(17/28).$$

We can now determine information such as

$$T(1) = 44 + 28e^{k} = 54.32°,$$

or solve for t in the following

$$50 = T(t) = 44 + 28e^{kt},\ t = \frac{\ln(3/14)}{k} = 1.54 \text{ hours.}$$

The logistic equation

The differential equation

$$\frac{dP}{dt} = kP(1 - P/M),\ P(0) = P_0,$$

is the logistic equation, a model of population growth that is controlled by the relative growth rate k and the carrying capacity M. This equation is separable, with solution

$$P(t) = \frac{M}{1 + \frac{M-P_0}{P_0}e^{-kt}}.$$

1. Suppose 400 fish are in a pond whose carrying capacity is estimated as 10,000 fish, and that the population triples to 1200 fish in the next year.

   - Assuming logistic growth, find the population of fish $P(t)$ as a function of time.
   - How long does it take for the fish population to reach 5000?

   We have $M = 10000$, $P(0) = 400$ and using the information $P(1) = 1200$ (t in years) we obtain $k = \ln 3$, so that

   $$P(t) = \frac{10000}{1 + 24e^{-kt}}.$$

   With this model, it takes $t = 2.68$ years for the population to reach 5000.