Exponential growth and decay

The initial value problem

$$y' = ky, \; y(0) = y_0,$$

describes exponential growth or decay (depending on the sign of the relative growth rate k). This equation is separable, and we have the solution

$$y(x) = y_0 e^{kx}.$$

1. Radium 226 has a half-life of 1590 years. Let $m(t)$ denote the amount of radium after t years in a sample with $m(0) = 100$ g. Answer the following questions:
   - Find $m(t)$.
   - How much of the sample is radium after 1000 years?
   - How much time must pass until there are only 30 g of radium left in the sample?

2. Newton's law of cooling states that the rate of change of the temperature $T(t)$ of an object in a constant temperature environment of temperature T_s is proportional the the temperature difference:

$$\frac{dT}{dt} = k(T - T_s), \;\; k, \;\; T_s \text{ constants.}$$

- Solve the differential equation above.
- Suppose a can of pop at 72° is put into a refrigerator whose temperature is 44° and that after 30 minutes, the temperature of the can is 61°. Find the temperature of the can $T(t)$ as a function of time.
- What is the temperature of the can after one hour?
- How long must you wait until the temperature of the can is 50°?

The logistic equation

The differential equation

$$\frac{dP}{dt} = kP(1 - P/M),\ P(0) = P_0,$$

is the logistic equation, a model of population growth that is controlled by the relative growth rate k and the carrying capacity M. This equation is separable, with solution

$$P(t) = \frac{M}{1 + \frac{M - P_0}{P_0} e^{-kt}}.$$

1. Suppose 400 fish are in a pond whose carrying capacity is estimated as 10,000 fish, and that the population triples to 1200 fish in the next year.
   - Assuming logistic growth, find the population of fish $P(t)$ as a function of time.
   - How long does it take for the fish population to reach 5000?