Center of mass (centroid)

Conisder a region in the plane bounded by two functions $f(x) \geq g(x)$, for $a \leq x \leq b$, which we assume to have uniform density ρ mass/area. Its moment around the x- and y-axes is given by

$$M_y = \rho \int_a^b x(f(x) - g(x))dx, \; M_y = \rho \int_a^b \frac{f(x)+g(x)}{2}(f(x)-g(x))dx = \frac{\rho}{2}\int_a^b (f(x)^2 - g(x)^2)dx.$$

[The moment of a (point) mass about an axis is the mass times the distance to the axis, $\mu = mx$.]

In the above situation, the moment of a thin vertical rectangle of width dx and height $f(x) - g(x)$ around the y-axis is given by the product of the distance x to the y-axis times the mass of the rectangle $\rho(f(x) - g(x))dx$. [Draw a picture.]

In the above situation, the momen of a thin vertical rectangle of width dx and height $f(x) - g(x)$ around the x-axis is given by the product of the distance from the center of mass of the rectangle to the x-axis, $\frac{1}{2}(f(x) + g(x))$, with the mass of the rectangle $\rho(f(x) - g(x))dx$. [Draw a picture.]

The center of mass $(\bar{x}, \bar{y})$ of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

where M is the total mass of the lamina

$$M = \rho \int_a^b (f(x) - g(x))dx.$$

Note that the centroid is independent of the mass density ρ (because it is uniform).

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1. Find the centroid of the region bounded by

$$y = x^2, \; y = \sqrt{x}.$$

2. Find the centroid of the region bounded by

$$y = x^2, \; y = 0, \; -1 \leq x \leq 1.$$

3. Find the centroid of the region bounded by

$$y = \cos x, \; y = \sin x, \; \pi/4 \leq x \leq 3\pi/4.$$

4. Find the centroid of the region bounded by

$$y = 1/x^3, \; y = 0, \; 1 \leq x < \infty.$$

Answers (I hope): 1. $(9/20, 9/20)$, 2. $(0, 3/10)$, 3. $(1 + \pi/4, 1/2\sqrt{2})$, 4. $(2, 1/5)$.