A very quick introduction to minimal dynamical systems

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Topological dynamical systems

A dynamical system (X, Γ) consists of:

- An infinite compact metric space X
- A topological group Γ
- A continuous group homomorphism

$$\Gamma \to \mathrm{Homeo}(X), \quad g \mapsto (x \mapsto gx)$$

Integer systems

Given a homeomorphism $\varphi : X \to X$ we define an action of $\mathbb{Z}$ on X by defining

$$n \,.\, x := \varphi^n(x) \text{ for every } n \in \mathbb{Z}, x \in X.$$

Conversely, when $\Gamma = \mathbb{Z}$, the system is generated by a single homeomorphism $\varphi : X \to X$ given by

$$\varphi(x) := 1.x \text{ for every } x \in X.$$

Thus for a $\mathbb{Z}$ system, we denote $(X, \mathbb{Z})$ by (X, φ) where φ is the generating homeomorphism.

Orbits

Let X be a compact metric space and $\varphi : X \to X$ a homeomorphism. As with any of the dynamical systems you've already seen, we would like to know about the orbits of points in (X, φ).

Given a point $x \in X$ we denote its orbit by

$$\mathrm{orb}_{\varphi}(x) := \{\varphi^n(x) \mid n \in \mathbb{Z}\}.$$

Note that sometimes in the literature the orbit of x is defined to be $\{\varphi^n(x) \mid n \in \mathbb{N}\}$. Here we will refer to this set as the forward orbit of x.

Similarly, we have the backward orbit of x given by $\{\varphi^{-n}(x) \mid n \in \mathbb{N}\}$.

Notions of recurrence

A point $x \in X$ is

- fixed if $\varphi(x) = x$,
- periodic if there exists $n \in \mathbb{Z}$ such that $\varphi^n(x) = x$,
- aperiodic if $\varphi^n(x) \neq x$ for every $n \in \mathbb{Z} \backslash \{0\}$,
- recurrent if for every open neighborhood U containing x there exists $n \in \mathbb{Z} \backslash \{0\}$ such that $\varphi^n(x) \subset U$,
- almost periodic if for every open neighborhood U there exists $N \in \mathbb{N}$ such that $\{m, m+1, \ldots, m+N-1\} \cap \{n \geq 0 \mid \varphi^n(x) \subset U\} \neq \varnothing$,
- transitive if $\mathrm{orb}_\varphi(x)$ is dense in X.

Silly examples

A very simple example of a dynamical system is (X, id). There, every point is a fixed point (and a periodic point, and a recurrent point, and almost periodic)

Another very simple example is to take $X = \{0,1,2,3,\ldots,n\}$ and the map $\sigma(j) = j+1 \mod n$. Here, every point is clearly periodic (and recurrent, and almost periodic, and even transitive!)

Unfortunately, these are not particularly interesting: in the first case, because nothing is happening, and in the second, because X is a finite set.

Let X be an infinite compact metric space. We would like to investigate dynamical systems that don't have proper sub-dynamical systems.

In particular, we cannot have either fixed or periodic points.

But we can try to ask for the next best thing: that ever point is almost periodic.

Minimal dynamical systems

We say a topological dynamical system (X, Γ) is minimal if whenever $E \subset X$ is a non-empty closed subset satisfying $\Gamma E \subset E$, then $E = X$.

Since we are interested in $\mathbb{Z}$ actions generated by a homeomorphism φ this means that there are no non-empty proper, closed φ-invariant subsets.

If (X, φ) is minimal, then we call φ a minimal homeomorphism.

Minimality of φ^{-1}

The following is an easy outcome of the definition, but is nevertheless quite useful.

PROPOSITION: Let (X, φ) be a dynamical system. Then (X, φ) is minimal if and only if (X, φ^{-1}) is minimal.

PROOF. It suffices to prove only one direction. So suppose that (X, φ) is minimal. Let $E \subset X$ be a non-empty closed subset and suppose that $\varphi^{-1}(E) \subset E$. Then $E = \varphi(\varphi^{-1}(E)) \subset \varphi(E)$ so $E = X$. Thus (X, φ^{-1}) is minimal.

Orbits in minimal systems

Proposition. Let (X, φ) be a dynamical system. Then the following are equivalent:

1. (X, φ) is minimal.
2. For every $x \in X$, its orbit $\mathrm{orb}_\varphi(x) = \{\varphi^n(x) \mid n \in \mathbb{Z}\}$ is dense in X. (That is, x is transitive.)
3. For every $x \in X$, its forward orbit $\{\varphi^n(x) \mid n \in \mathbb{N}\}$ is dense in X.
4. For every $x \in X$, its backward orbit $\{\varphi^{-n}(x) \mid n \in \mathbb{N}\}$ is dense in X.

PROOF. $1 \implies 2$ Suppose that φ is minimal and let $x \in X$. Then $E = \overline{\{\varphi^n(x) \mid n \in \mathbb{Z}\}}$ is a non-empty closed subset of X. Let $y \in E$ and let $(y_n)_{n\in\mathbb{N}} \subset \{\varphi^n(x) \mid n \in \mathbb{Z}\}$ be a sequence converging to y. Then there is $(m_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ such that $y_n = \varphi^{m_n}(x)$ for every $n \in \mathbb{N}$. It follows that $\varphi(y) = \lim_{n\to\infty} \varphi(\varphi^{m_n}(x)) = \lim_{n\to\infty} \varphi^{m_n+1}(x) \subset E$. Thus $\varphi(E) \subset E$, so $E = X$.

That $1 \implies 3$ is identical after replacing the set E with $F = \overline{\{\varphi^n(x) \mid n \in \mathbb{N}\}}$.

We have $3 \iff 4$ because φ^{-1} is also minimal.

That $3,4 \implies 2$ is trivial.

$3 \implies 1$: Let $E \subset X$ be a non-empty closed subset with $\varphi(E) \subset E$. Then for every $x \in E$, we have $\{\varphi^n(x) \mid n \in \mathbb{N}\} \subset E$ and hence $\overline{\{\varphi^n(x) \mid n \in \mathbb{N}\}} = X \subset E$. Thus $E = X$ so (X, φ) is minimal.

$2 \implies 3$: Let $x \in X$ and let $U \subset X$ be a non-empty open set. Since orbits are dense, $\varphi^n(U), n \in \mathbb{Z}$ is an open cover of X. By compactness of X, there exists $N \in \mathbb{N}$ such that $\varphi^n(U), -N \leq n \leq N$ cover X. Since φ is a homeomorphism, we have $\varphi^{-N}(X) = X$, which means that $\varphi^n(U), -2N \leq n \leq 0$ cover X. In particular $x \in \varphi^n(U)$ for some $n \leq 0$, so $\varphi^{-n}(x) \subset U$. Thus every non-empty open set $U \subset X$ is in the forward orbit of x, which is to say, $\{\varphi^n(x) \mid n \in \mathbb{N}\}$ is dense. $\blacksquare$

Return times and almost periodicity

Let (X, φ) be a minimal dynamical system with X an infinite compact metric space.

Let $E \subset X$ be closed subset with non-empty interior.

For any $x \in E$ we define its first return time $r_E(x)$ to E to be the number

$$r_E(x) := \min\{n > 0 \mid \varphi^n(x) \in E\}$$

Return times and almost periodicity

Note that for any such set E there are only finitely many return times.

Indeed, at every point $x \in E$, there is a small neighbourhood U_x of x such that $r_E(x) = r_E(y)$ for every $y \in U_x$.

Since E is compact and $\{U_x \mid x \in E\}$ cover E, there are $x_1, \ldots, x_k \in E$ such that $U_{x_1}, \ldots, U_{x_k}$ also cover E.

Thus for every $x \in E$ there is some $1 \leq j \leq k$ with $r_E(x) = r_E(x_j)$

Return times and almost periodicity

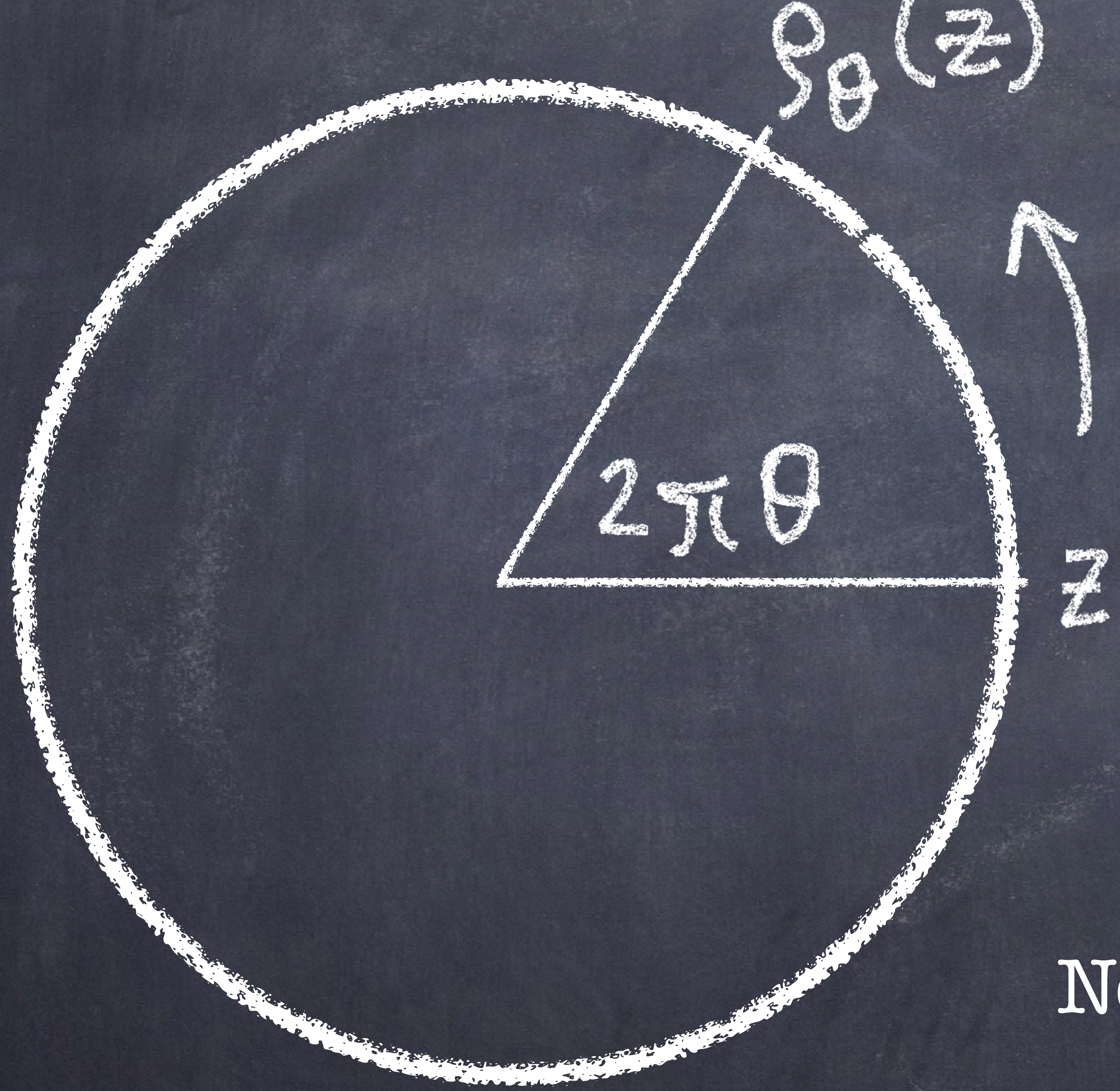
After this observation, it is not hard to show that if (X, φ) is a minimal dynamical system with X an infinite compact metric space, then every $x \in X$ is almost periodic:

- for every open neighborhood U there exists $N \in \mathbb{N}$ such that $\{m, m+1, \ldots, m+N-1\} \cap \{n \geq 0 \mid \varphi^n(x) \subset U\} \neq \emptyset$

Note that having every point almost periodic is not equivalent to minimality.

It is true, however, that $(\overline{\mathrm{orb}_\varphi(x)}, \varphi|_{\overline{\mathrm{orb}_\varphi(x)}})$ is minimal whenever x is almost periodic.

Rotations of the circle



Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

Then, for any θ with $0 \leq \theta < 1$, the map

$$\rho_\theta : S^1 \to S^1$$

defined by

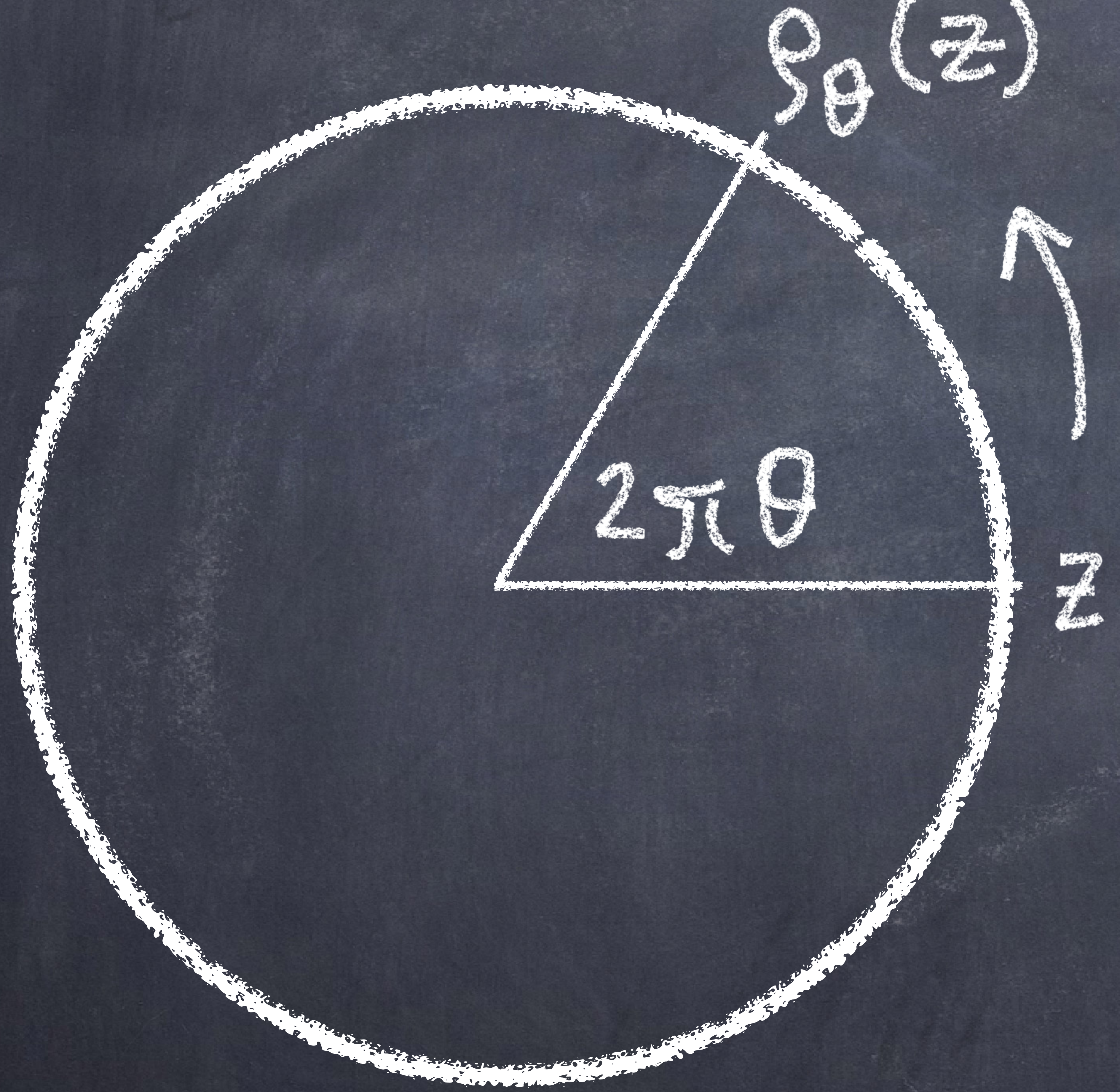
$$\rho_\theta(z) = e^{2\pi i\theta} z$$

is a homeomorphism.

Note that we can also define ρ_θ as follows:

Identify $S^1 \cong \mathbb{R}/\mathbb{Z}$ and let $\rho_\theta(z) = z + \theta \mod \mathbb{Z}$.

Rotations of the circle



Suppose that $\theta = \dfrac{a}{b} \in \mathbb{Q}$.

Then $\rho_\theta^b(z) = e^{2\pi bi}z = e^{2\pi i}z = z$ for every $z \in S^1$. Thus every point is periodic.

On the other hand, if $\theta \notin \mathbb{Q}$, there are no periodic points.

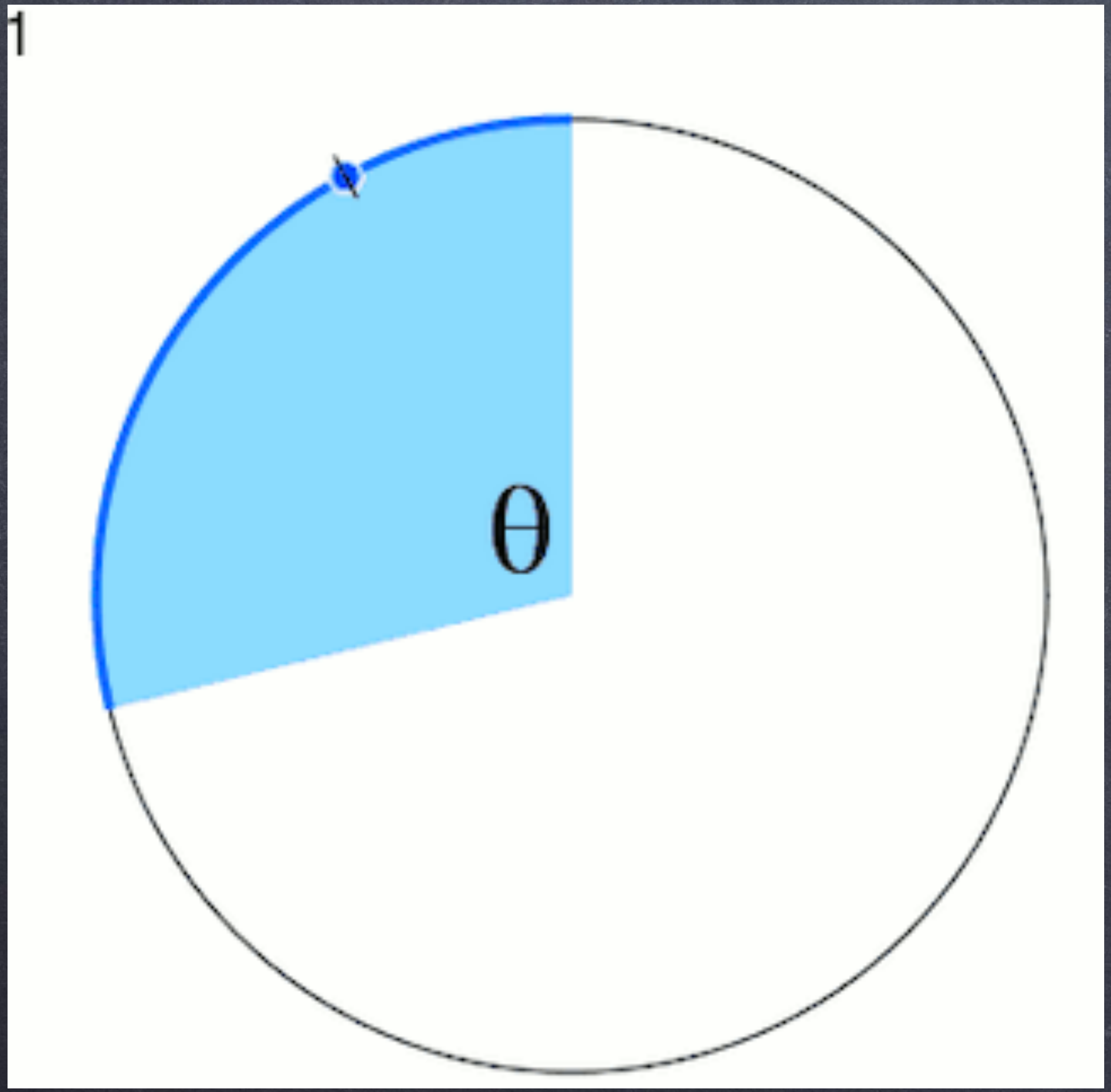
Indeed, in this case (S^1, ρ_θ) is minimal.

Idea: Show that the orbit of every point meets every open arc/interval.

For $\theta \in [0,1)$, define $T_\theta : [0,1) \to [0,1)$ by $t \mapsto t + \theta \bmod 1$. For $x \in [0,1)$ define the θ-coding of x to be the sequence (x_n) where

$$x_n = \begin{cases} 1 & \text{if } T_\theta^n(x) \in [0,\theta), \\ 0 & \text{else.} \end{cases}$$

Let w be an infinite sequence of 0s and 1s. The sequence w is Sturmian if for some $x \in [0,1)$ and some irrational $\theta \in (0,\infty)$, w is the θ-coding of x.



“Sturmian sequence generated by an irrational rotation with $\theta = 0.2882748715208621$ and $x = 0.078943143$”

from https://commons.wikimedia.org/wiki/File:Sturmian-sequence-from-irrational-rotation.gif

Odometers

Fix a non-zero natural number $n > 1$ and let $X = \{0,1,\ldots,n-1\}^{\mathbb{N}}$, the set of bi-infinite sequences with entries in $\{0,\ldots,n-1\}$. Given $\{0,\ldots,n-1\}$ the discrete topology and X the product topology.

Then X is a non-empty, compact, metrisable, totally disconnected and has no isolated points. That is, X is the Cantor set.

Odometers

We define a homeomorphism of X as follows.

If $x = (x_j)_{j \in \mathbb{N}}$ is the sequence such that $x_j = n - 1$ for every $j \in \mathbb{N}$, then

- $\varphi(x)_j = 0$ for every $j \in \mathbb{N}$.

Otherwise, for $x = (x_j)_{j \in \mathbb{N}}$, let $m > 0$ be the least integer such that $x_j = n - 1$ for every $j < m$. Then

- $\varphi(x)_j = 0$ for every $j < m$,
- $\varphi(x)_m = x_m + 1$, and
- $\varphi(x)_j = x_j$ for every $j > m$.

Odometers

Every odometer is minimal.

To see this, let us consider again the 2-odometer $(X = \{0,1\}^{\mathbb{N}}, \varphi)$.

To show the map is minimal, it is enough to show that given an arbitrary $x \in X$, and any open set $U \subset X$, there is some $j \in \mathbb{N}$ such that $\varphi^j(x) \subset U$.

Or, equivalently, for any $y \in X$ we can move x arbitrarily close to y.

For example, if y is the sequence of all zeros, the further we move x, the more times will come back to a sequence with more and more zeros on the left.

2-Odometer

0	0	0	0	0	0	0	0	0

2-Odometer

1	0	0	0	0	0	0	0	0

2-Odometer

0	1	0	0	0	0	0	0	0

2-Odometer

1	1	0	0	0	0	0	0	0

2-Odometer

0	0	1	0	0	0	0	0	0

2-Odometer

1	0	1	0	0	0	0	0	0

2-Odometer

1	1	1	0	0	0	0	0	0

2-Odometer

0	0	0	1	0	0	0	0	0

1	0	0	1	0	0	0	0	0

1	1	0	1	0	0	0	0	0

1 1 1 1 0 0 0 0 0

0	0	0	0	1	0	0	0	0

1	0	0	0	1	0	0	0	0

1	1	0	0	1	0	0	0	0

1	1	1	0	1	0	0	0	0

1	1	1	1	1	0	0	0	0

0	0	0	0	0	1	0	0	0

1 0 0 0 0 1 0 0 0

1	1	0	0	0	1	0	0	0

1	1	1	0	0	1	0	0	0

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1	1	1	1	1	1	0	0	1

1	1	1	1	1	1	1	0	1

1	1	1	1	1	1	1	1	1

0	0	0	0	0	0	0	0	0

Odometers

Of course, the length of the sequences are infinite, so we never actually return to the sequence of all zeros, despite coming close an infinite number of times.

Odometers are examples of distal systems. (X, φ) is distal if the following holds:

For every $x, y \in X$ there exists $\delta > 0$ such that $d(\varphi^n(x), \varphi^n(y)) \geq \delta$ for every $n \geq 0$.

Moreover, they are equicontinuous:

For every $\epsilon > 0$ there exists $\delta > 0$ such that $d(\varphi^j(x), \varphi^j(y)) < \epsilon$ for every $j \in \mathbb{Z}$ whenever $d(x, y) < \delta$.

Odometers

Let X be the Cantor set and $\varphi : X \to X$ a homeomorphism.

Suppose that (X, φ) is minimal and equicontinuous.

Then (X, φ) is conjugate to an odometer (Y, ψ).

There are, however, minimal Cantor systems which are not conjugate to odometers.

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More generally, a homeomorphism on any finite CW complex with non-zero Euler characteristic always has a periodic point. Hence such a space cannot admit a minimal homeomorphism.

Let (X, φ) be a topological dynamical system. Then X will always admit minimal subsets. That is to say, there exists $Y \subset X$ closed, compact, such that $(Y, \varphi|_Y)$ is a minimal dynamical system.

However, even if X is an interesting space, Y might not be. For example, the orbit of any periodic point is a minimal subset, but is finite.

In general we do not know how to characterise those compact metric spaces X which do admit minimal homeomorphisms.

Often, such construction take an known dynamical system and constructs a minimal homeomorphism along with the space.

Société Mathématique de France
Astérisque 49 (1977) p.37-59

EXISTENCE DE DIFFÉOMORPHISMES MINIMAUX

par

Albert FATHI et Michael R. HERMAN

THÉORÈME 1. Toute variété M^n compacte connexe qui admet une action C^∞ localement libre effective de T^1 admet un difféomorphisme de classe C^∞ (isotope à l'identité) qui est minimal.

THÉORÈME 3. Toute variété M^n admettant une action localement libre C^∞ de T^1 admet un difféomorphisme C^∞ strictement ergodique.

It follows from the work of Fathi and Herman, for example, that any odd-dimensional sphere S^d, with $d \geq 3$, admits a minimal homeomorphism (in fact, diffeomorphism).

From this, in work of Deeley, Putnam and myself, we were able to construct all sorts of bizarre spaces!

Theorem 1.5. *Let S^d be a sphere with odd dimension $d \geq 3$, and let $\varphi : S^d \to S^d$ be a minimal diffeomorphism. Then there exist an infinite compact metric space Z with covering dimension d or $d-1$ and a minimal homeomorphism $\zeta : Z \to Z$ satisfying the following.*

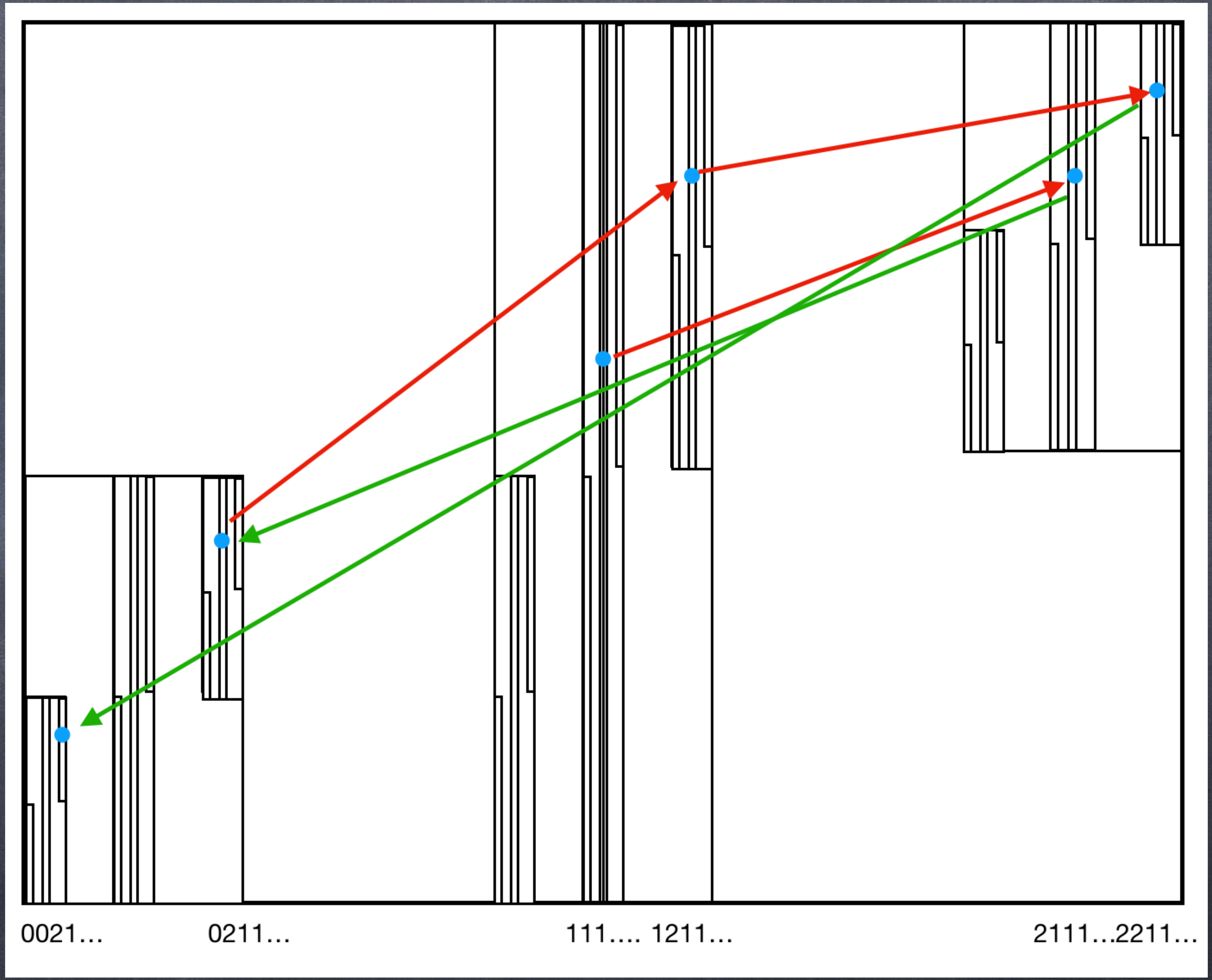
(1) *Z is compact, connected, and homeomorphic to an inverse limit of compact contractible metric spaces $(Z_n, d_n)_{n \in \mathbb{N}}$.*
(2) *For any continuous generalized cohomology theory we have an isomorphism $H^*(Z) \cong H^*(\{\mathrm{pt}\})$. In particular this holds for Čech cohomology and K-theory.*
(3) *There is an almost one-to-one factor map $q : Z \to S^d$ which induces a bijection between ζ-invariant Borel probability measures on Z and φ-invariant Borel probability measures on S^d.*

Fattening up Cantor

Fattening up Cantor

Fattening up 3-odometer by intervals

- Floyd 1949
- Gjerde–Johansen 1999

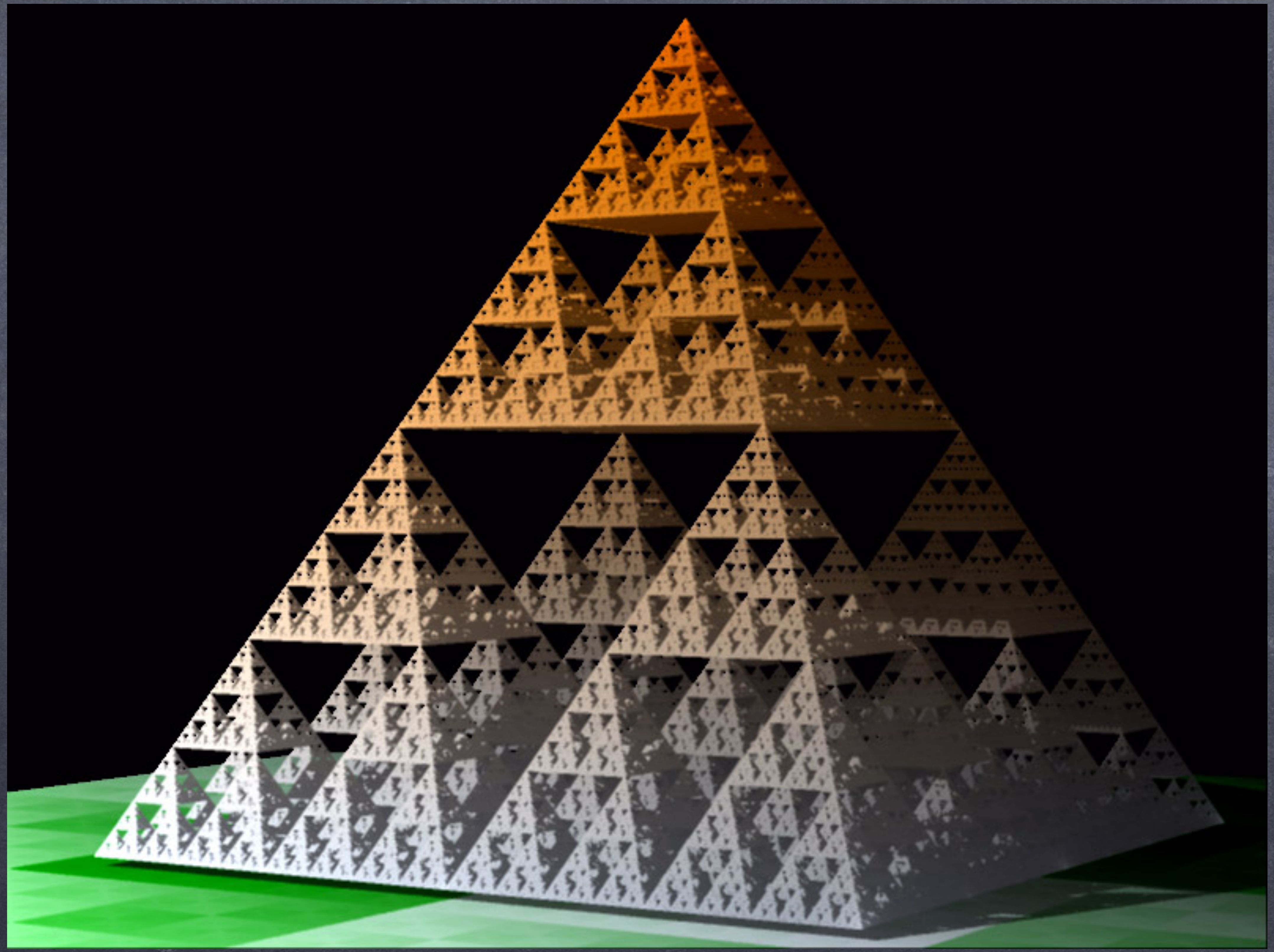


Further nonhomogeneous systems

In work with Deeley and Putnam, we further generalized these further by using iterated function systems.

Theorem 1.5. *Let $(C, d_C, \mathcal{F})$ be a compact, invertible iterated function system and let (X, φ) be a minimal homeomorphism of the Cantor set. There exists a minimal extension, $(\tilde{X}, \tilde{\varphi})$ of (X, φ) with factor map $\pi : (\tilde{X}, \tilde{\varphi}) \to (X, \varphi)$ such that, for each x in X, $\pi^{-1}\{x\}$ is a single point or is homeomorphic to C. Moreover, both possibilities occur.*

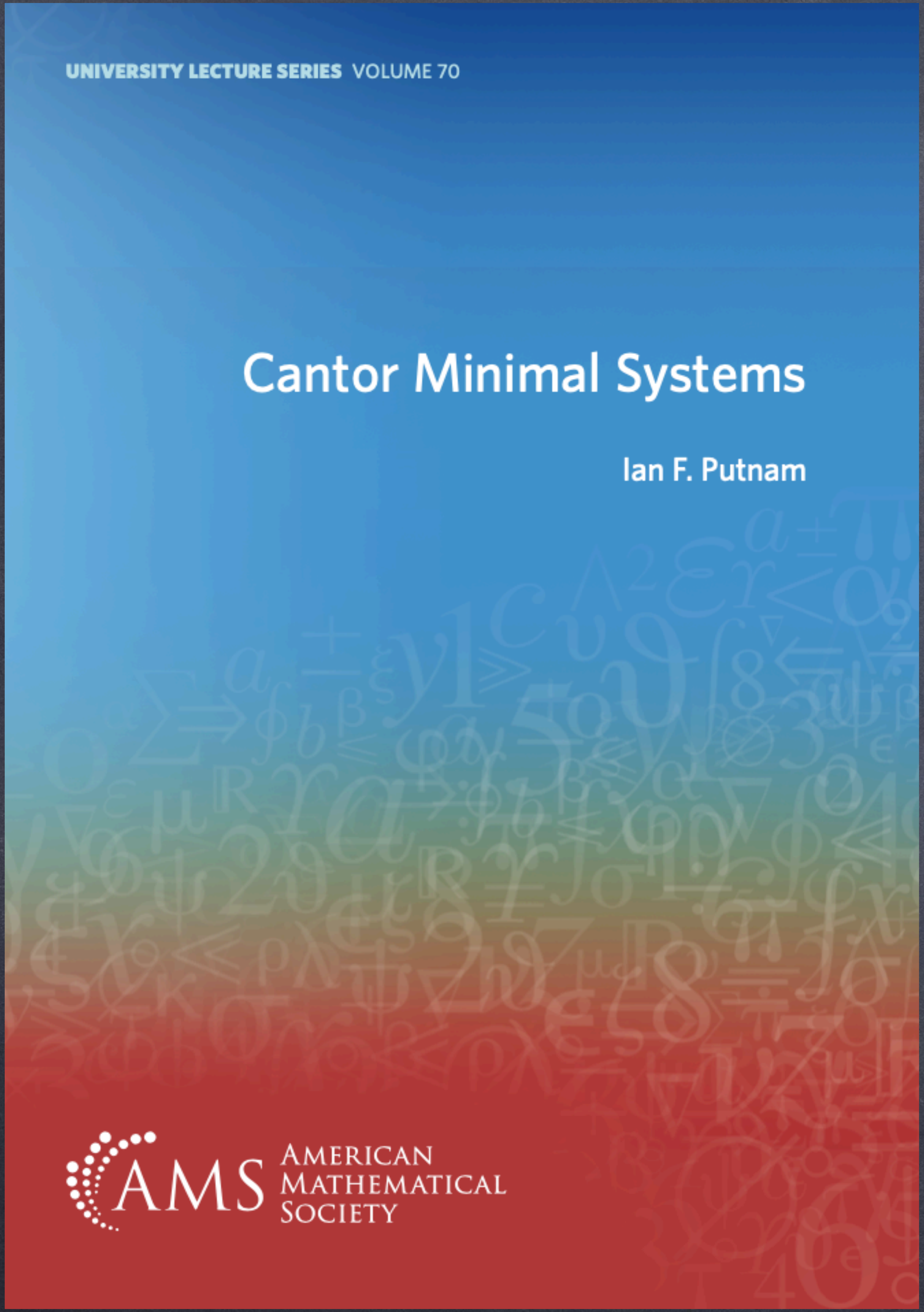
Now we get strange spaces that look like a Cantor set from one direction, and for example, the Sierpinski pyramid from another.



Things we didn't discuss

- Ergodic measures and the links to ergodic theory
- Entropy for minimal systems
- Notions such as mixing, weak mixing
- Invariants for minimal dynamical systems
- Links to other areas such as the theory of C^*-algebras and topological groupoids

...in other words, there's much more to discover!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Thanks for listening!