Splitting factor maps into u- and s-bijective maps.

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Brief history

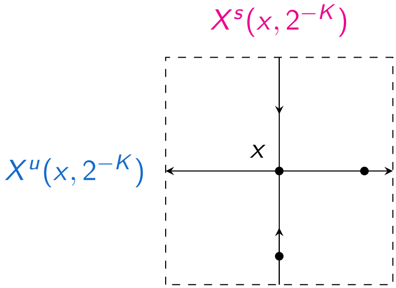
- Smale 1967 Structural stability & the horseshoe
- Anosov 1967 Globally hyperbolic systems
- Anosov 1967 Anosov diffeomorphisms
- Smale 1967 Definition of Axiom A
- Ruelle Definition of Smale space

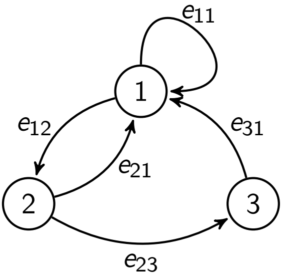
Intuitive description

A Smale space is a hyperbolic dynamical system (X, ϕ) where X is a compact metric space and ϕ is a homeomorphism.

$$\text{Hyperbolicity} \implies \text{local product structure}$$

i.e x in X given by local expanding and contracting directions.





$$\sigma(\ldots, e_{12}, e_{23}, e_{31}.e_{11}, e_{11}, e_{11}, \ldots) = (\ldots, e_{12}, e_{23}, e_{31}, e_{11}.e_{11}, e_{11}, \ldots)$$

Shifts of finite type

Let G be a finite directed graph which consists of a vertex set G^0, an edge set G^1, and two maps $r, s : G^1 \to G^0$. The source vertex of edge e is given by $s(e)$ and the range vertex is given by $r(e)$.

Definition

We define

$$\Sigma_G = \{(e_n)_{n\in\mathbb{Z}} \mid e_n \in G^1,\ r(e_n) = s(e_{n+1}) \text{ for all } n \text{ in } \mathbb{Z}\}$$

With the left shift map $\sigma : \Sigma_G \to \Sigma_G$,

$$\sigma(x)_n = x_{e+1}.$$

Hyperbolic toral automorphism

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Define $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ by

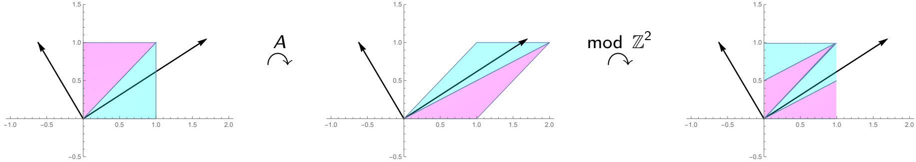
$$f_A([x]) = [Ax]$$

where x is in $\mathbb{R}^2$ and $[x]$ denotes its equivalence class in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. By the integer components and the determinant, f_A is an invertible map.

Eigenvalues : γ and $-\gamma^{-1}$, where $\gamma = \frac{1+\sqrt{5}}{2} > 1$.

A is *hyperbolic* $\sim$ none of its e-values lie on the unit circle.

Eigenvectors: $v_u = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ and $v_s = \begin{bmatrix} -\gamma^{-1} \\ 1 \end{bmatrix}$.



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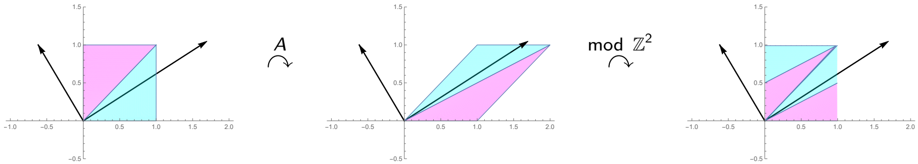
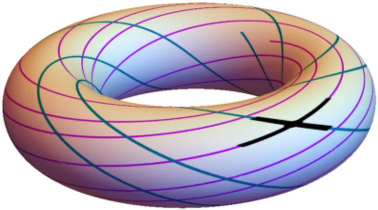
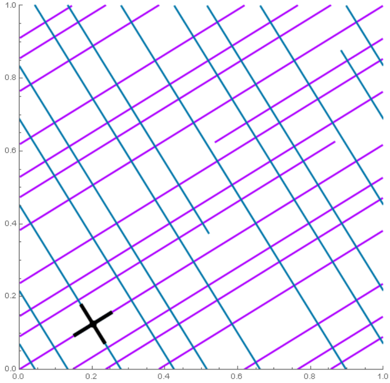


Notice $\mathbb{R}^2 = \{tv_u \mid t \in \mathbb{R}\} \oplus \{tv_s \mid t \in \mathbb{R}\} = E^u \oplus E^s$

On a HTA the global unstable and stable sets wrap around densely.

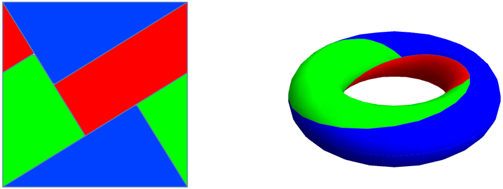




The local stable and unstable sets are given by moving a little bit along the eigendirections. Locally, $\mathbb{T}^2$ can be viewed as $\mathbb{R} \times \mathbb{R}$.

The HTA can be modeled using symbolic dynamics by way of Markov partitions, where $\pi : (\Sigma_G, \sigma) \to (\mathbb{T}^n, A)$ is a finite-to-one factor map.

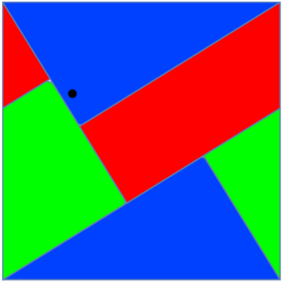
- Berg 1967 dimension $d = 2$.
- Adler and Weiss 1967 extended and formalized for $d = 2$.
- Sinai 1968 any finite dimension d.
- Bowen 1970, for Smale spaces.



How do these partitions work

We would like to construct a symbolic representation for the dynamical system $(\mathbb{T}^2, A)$ and a map π.

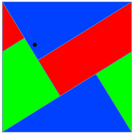
Let x be in $\mathbb{T}^2$, how can we create a coding for this element?



Track the orbits of x.

Each rectangle corresponds to a vertex on the graph G.

$A^0(x)$



B

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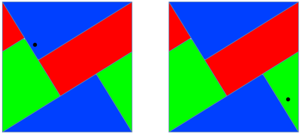
$A^0(x)$ $A^1(x)$



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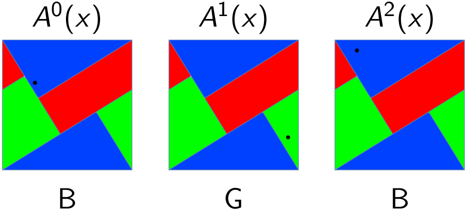

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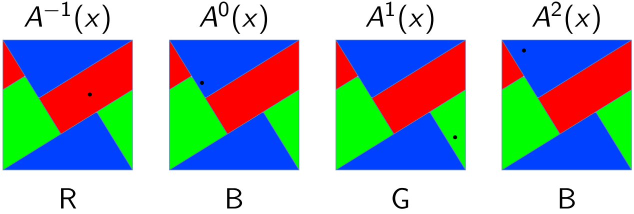

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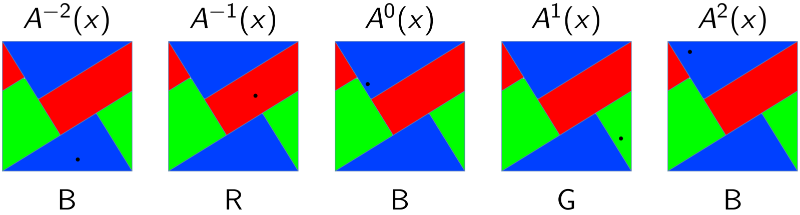
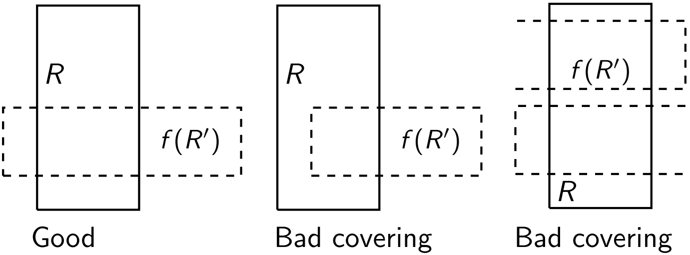


Markov Property



Theorem (Bowen)

If (X, f) is an irreducible Smale space then (X, f) has Markov partitions. Equivalently, there is a shift of finite type, (Σ, σ) and a factor map $\pi : \Sigma \to X$.

Fact

We say that $\pi : (X, f) \to (Y, g)$ *is an s-bijective map if for every* x *in* X*, the map* $\pi : X^s(x, \epsilon) \to Y^s(\pi(x), \epsilon')$ *is a local homeomorphism.*

A u-bijective map is defined and characterized analogously.

Given $(\mathbb{T}^d, A)$, we can find a factor map π.

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where $m + n = d$.

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Note: This map cannot be s-bijective nor u-bijective.

Definition

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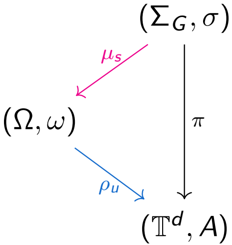
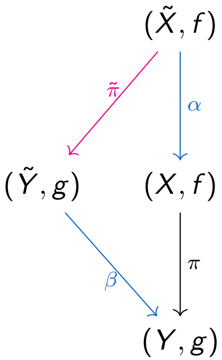


Theorem (Putnam)

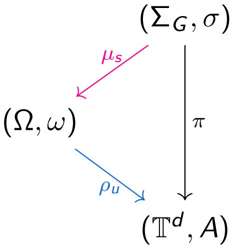
Let (X, d_x, f) *and* (Y, d_Y, g) *be irreducible Smale spaces and suppose that*

$$\pi : (X, f) \to (Y, g)$$

is an almost one-to-one factor map. Then there exist irreducible Smale spaces, $(\tilde{X}, f)$ *and* $(\tilde{Y}, g)$ *and factor maps* $\alpha, \beta, \tilde{\pi}$ *as shown. Moreover, the diagram is commutative,* α *and* β *are u-resolving and* $\tilde{\pi}$ *is s-resolving.*



Suppose we have a splitting where μ_s, is an s-bijective map and ρ_u, a u-bijective map with a commutative diagram,



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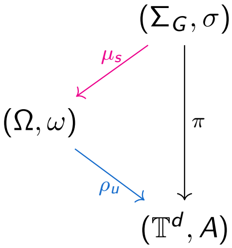

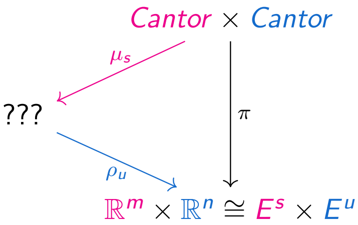

What must ??? look like locally?

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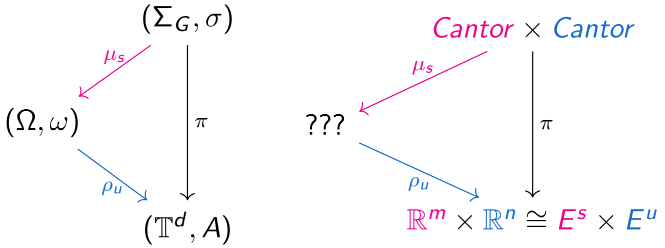
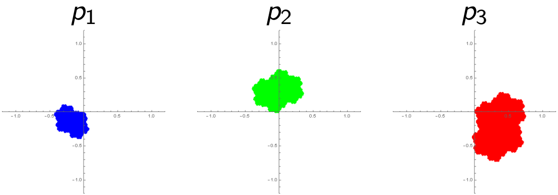


What must ??? look like locally? $Cantor \times \mathbb{R}^n$
What is a candidate space for ??? ?

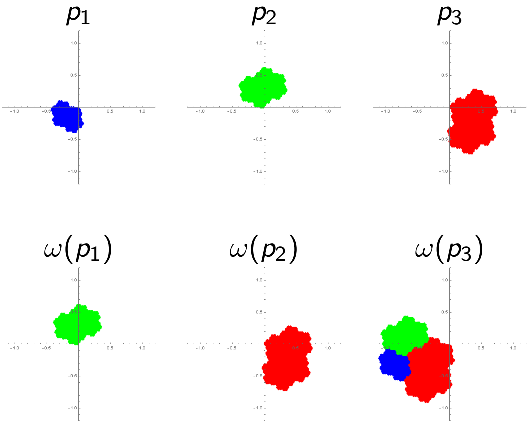
3rd Example: Substitution tiling systems, $(\Omega, \mathcal{P}, \omega)$

Prototiles, $\mathcal{P} = \{p_1, p_2 \ldots, p_n\}$. Each $p_i \subseteq \mathbb{R}^d$ is the closure of its interior.

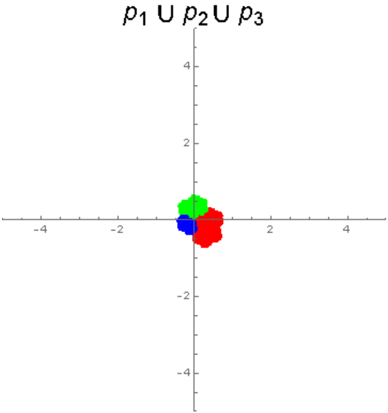


A tile t is a translation of some prototile.

A substitution rule $\omega(p_i)$ that inflates, possibly rotates and subdivides with translates of prototiles.



A partial tiling is a collection of tiles whose interiors are pairwise disjoint.
A tiling is a partial tiling whose union is $\mathbb{R}^d$.



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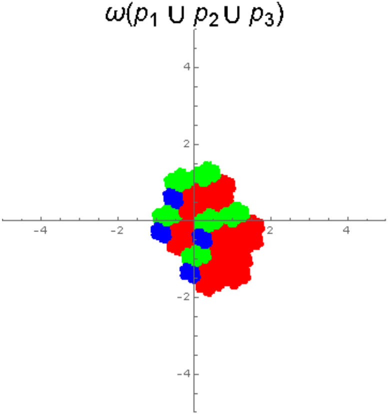

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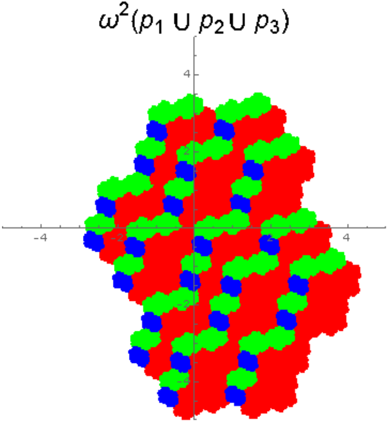
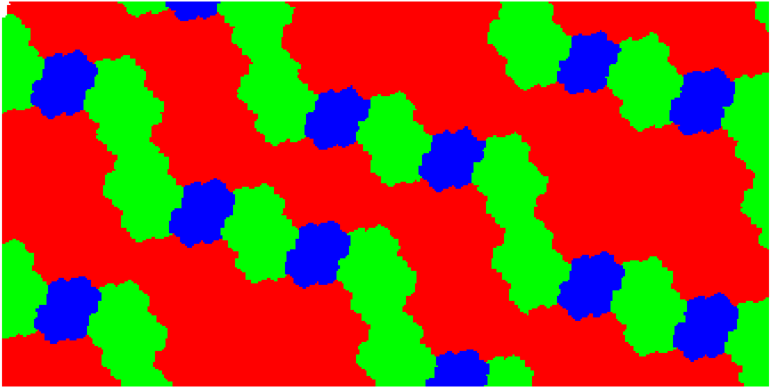


We define Ω to be the set of tilings that contain the patches of T.



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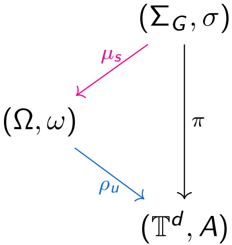


Thesis questions

1. Given a factor map from SFT to Smale space, is there a condition on if it splits?
2. What is the simplest SFT to use as a model? Can we find a factor map for such systems? Can we find a splitting for such systems?

Focus: Consider an HTA as our Smale space.

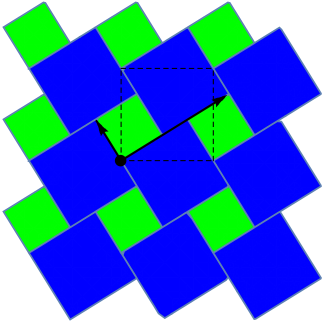
Given a factor map from a SFT to a HTA, how can we determine if a splitting exists?

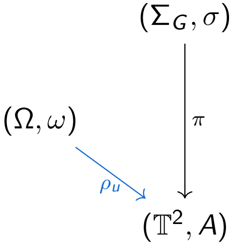
Theorem (B, Putnam)

If a splitting for the factor map $\pi : \Sigma \to \mathbb{T}^d$ exists then it must satisfy Condition A.

(Condition A is technical, and it is not necessary to be written explicitly)

Example for when a splitting does not exist, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$





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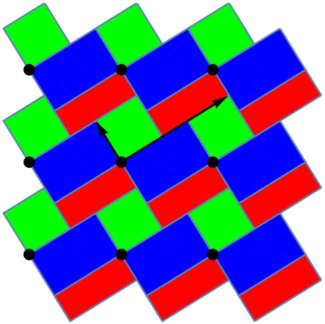
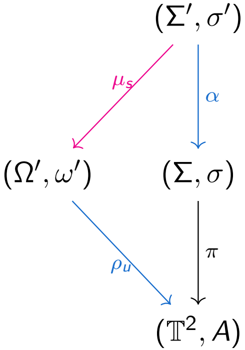


Due to work of Wieler 2005.



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1. Given a factor map from SFT to a Smale space, is there a condition on if it splits?
2. What is the simplest SFT to use as a model?
   Can we find a splitting for such systems?

In the 2×2 case, for which HTAs does a splitting exist with the a factor map from a SFT defined by the same matrix?

Theorem (B, Putnam)

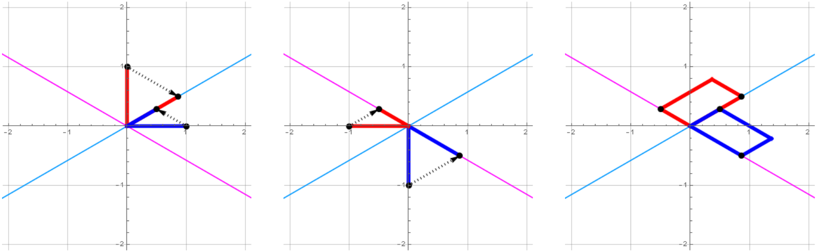
If A is hyperbolic, with $\det(A) = 1$, positive entries and is not $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ then there exists a factor map from a SFT given by A which has a splitting.

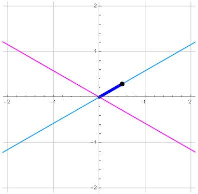
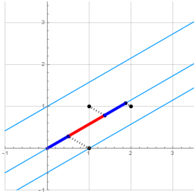
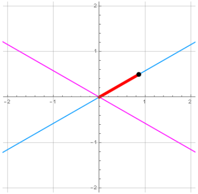
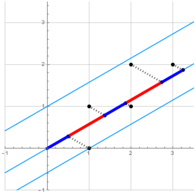
Outline of proof

1. Define a new way to construct Markov partitions
2. Use the MPs to define tiling spaces with some very nice properties
3. Construct maps between the SFT, tiling space and HTA.
4. Identify for which points the maps are 2-to-1 and 1-to-1.
5. Conclude that a splitting exists

Markov partitions are given by projecting basis vectors onto the two eigenspaces and then ”summing” the sets together.

Onto E^u | Onto E^s | Markov partition

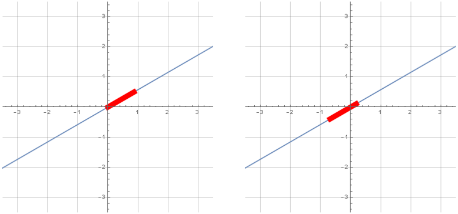


Tile	Substitution
	
	

From the SFT to the tiling space, $\mu_s : \Sigma \to \Omega$

$$x = (\dots e_{-m}, \dots e_{-1}, \overbrace{e_0,}^{tile} \overbrace{e_1, \dots e_n \dots}^{origin})$$

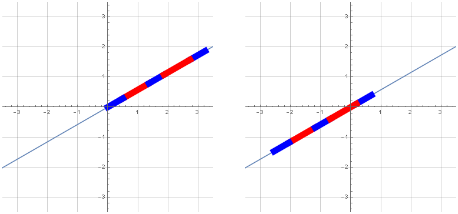
$$T_0(x) = R^u_{r(e_0)} - \sum_{m=1}^{\infty} A^{-m} \nu(e_m)$$



From the SFT to the tiling space, $\mu_s : \Sigma \to \Omega$

$$x = (\dots e_{-m}, \dots \overbrace{e_{-1}}^{patch}, \overbrace{e_0, e_1, \dots e_n \dots}^{shift})$$

$$T_1(x) = \omega(R^u_{r(x_{-1})}) - \sum_{m=-1}^{\infty} A^{-m}\nu(x_m)$$

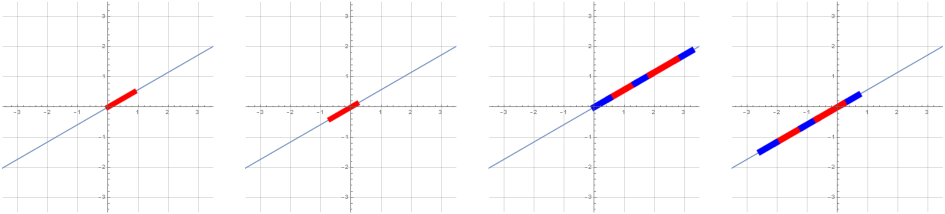


From the SFT to the tiling space, $\mu_s : \Sigma \to \Omega$

$$x = (\overbrace{\dots e_{-m}, \dots e_{-1}, e_0,}^{\textit{tiling}} \overbrace{e_1, \dots e_n \dots}^{\textit{origin}})$$

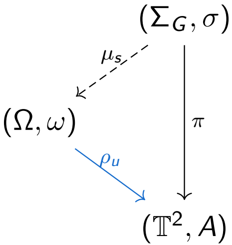
$$T_n(x) = \omega^n(R^u_{r(x_{-n})}) - \sum_{m=-n}^{\infty} A^{-m}\nu(x_m)$$

$$T(x) = \bigcup_{n=1}^{\infty} T_n(x)$$

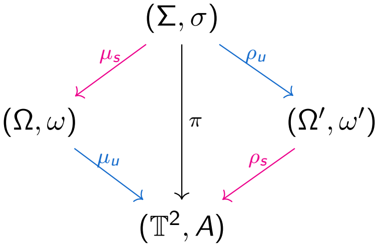


Tiling space to the SFT, $\rho_u : \Omega \to \mathbb{T}^2$

$$\pi \circ \mu_s^{-1}$$



Full splitting

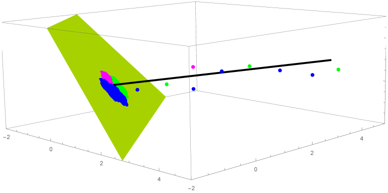


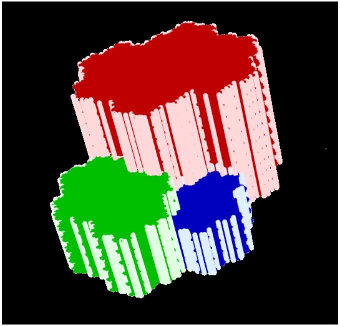
Many questions

1. What's going on with $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$? Or hyperbolic toral automorphisms with det -1?
2. What about for dimension 3 or above?
3. We focused on HTAs, can this work be generalized to other Smale spaces?
4. When does our Markov partition construction produce regular sets?

Let $f_A : \mathbb{T}^3 \to \mathbb{T}^3$ be given by $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

The Markov partition is given by the following (viewed in $\mathbb{R}^3$).





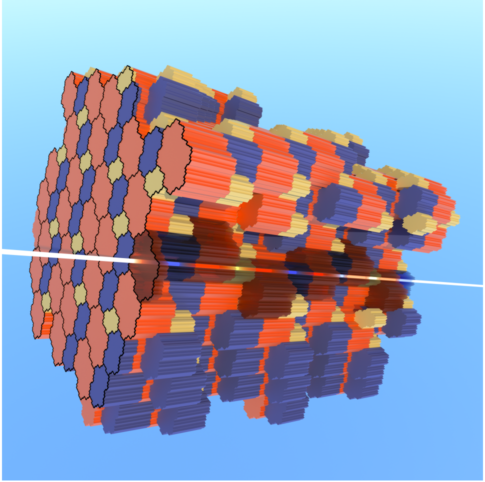


Image created by Edmund O. Harriss

"Of course the most rewarding part is the 'Aha' moment, the excitement of discovery and enjoyment of understanding something new- the feeling of being on top of a hill and having a clear view. But most of the time, doing mathematics for me is like being on a long hike with no trail and no end in sight." -Maryam Mirzakhani

Thank you for your attention!

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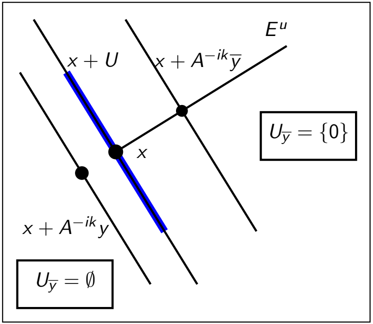
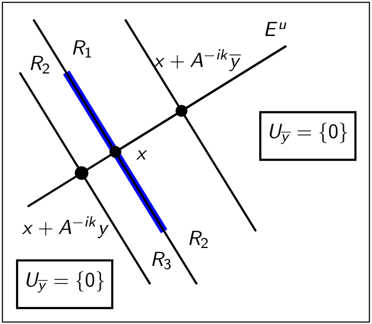
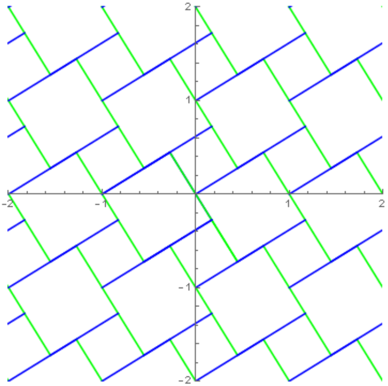


Condition A fails.



Condition A is satisfied.

The boundary $\partial\mathcal{R} = \partial^s\mathcal{R} \cup \partial^u\mathcal{R}$



Our example does not satisfy Condition A.