

# Hilbert $C^*$ -modules (Lance)

Overview: Hilbert  $C^*$ -module is kind of a generalization of a Hilbert space:

- have a  $\mathbb{C}$ -valued inner product
- $\mathbb{C}$  acts on vectors

replace  $\mathbb{C}$  by an arbitrary  $C^*$ -algebra

Def: Let  $A$  be a  $C^*$ -alg. An inner product

$A$ -module  $E$  is a complex vector space

such that  $\lambda(xa) = (\lambda x)a = x(\lambda a)$   
 $\mathbb{C} \quad E \quad A$

with an  $A$ -valued inner-product:

$$\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$$

$$\langle y, xa \rangle = \langle y, x \rangle a$$

$$\langle y, x \rangle = \langle x, y \rangle^*$$

$$\langle x, x \rangle \geq 0 \quad ; \quad \langle x, x \rangle = 0 \Rightarrow x = 0$$

prop (1.1)  
(Cauchy-Schwarz)

$$\langle y, x \rangle \langle x, y \rangle \leq \underbrace{\|\langle x, x \rangle\|_A}_{\text{norm on } A} \underbrace{\langle y, y \rangle}_{\text{"} \langle y, y \rangle^* \text{"}}$$

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Can also make a norm on  $E$ :

$$\|x\|_E = \|\langle x, x \rangle\|_A^{1/2}$$

prop 1.1

$\Rightarrow$

$$\|\langle x, y \rangle\|_A \leq \|x\|_E \|y\|_E \quad \begin{array}{l} x, y \in E \\ \langle x, y \rangle \in A \end{array}$$

Can also make an absolute value on  $E$  ( $A$ -valued)

$$|x|_E = \underbrace{\langle x, x \rangle^{1/2}}_{\text{positive element in } A}$$

prop 1.1

$\Rightarrow$

$$|\langle x, y \rangle|_A \leq \|x\|_E |y|_E$$

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$$|a| = \sqrt{a^* a}$$

Def: An inner-prod  $A$ -module which is complete with respect to its norm is called a Hilbert  $A$ -module (or a Hilbert  $C^*$ -module over  $A$ ).

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EX:

- Hilbert  $\mathbb{C}$ -modules are Hilbert spaces
- a  $C^*$ -algebra  $A$  acting on itself is a Hilbert  $A$ -module:

$$x \cdot a = xa$$

$$\langle x, y \rangle = x^* y$$

- $M_{m \times n}(\mathbb{C})$  is a Hilbert  $M_n(\mathbb{C})$ -Modules

$$\begin{bmatrix} \phantom{x} \\ \phantom{x} \\ \phantom{x} \end{bmatrix}_{m \times n} \begin{bmatrix} \phantom{x} \\ \phantom{x} \\ \phantom{x} \end{bmatrix}_{n \times n} = \begin{bmatrix} \phantom{x} \\ \phantom{x} \\ \phantom{x} \end{bmatrix}_{m \times n}$$

$$x \cdot a = xa$$

$$\begin{aligned} \langle x, y \rangle &= x^* y \\ &= \bar{x}^T y \\ &\in M_n(\mathbb{C}) \end{aligned}$$

$$\bigoplus_{i=1}^{\infty} E_i = \left\{ (x_i)_{i=1}^{\infty} \mid \sum_{i=1}^{\infty} \langle x_i, x_i \rangle \text{ converges in } A \right\}$$

each  $E_i$  is a  
Hilbert  $A$ -module

if each  $E_i = A$ , then  $H_A = \bigoplus_{i=1}^{\infty} A$ .

e.g. Let  $A = C([0,1])$ ,  $F = \{f \in A \mid f(0) = 0\} \subseteq A$

By definition  $F^{\perp} = \{g \in A \mid \langle f, g \rangle = 0 \ \forall f \in F\}$

$\langle f, g \rangle(t) = f^*(t) \cdot g(t)$  so  $F^{\perp} = \{0\}$

then  $F^{\perp\perp} = A$ .

Def: A Hilbert  $A$ -module  $E$  is called full if  
the ideal

$$I = \text{span} \{ \langle x, y \rangle_E \mid x, y \in E \} \subseteq A$$

is dense in  $A$ .

# Maps between Hilbert $C^*$ -modules

Let  $E, F$  be Hilbert  $A$ -modules.

Define  $L(E, F)$  to be the set of all maps  $t: E \rightarrow F$  for which there is a map  $t^*: F \rightarrow E$  such that

$$\langle tx, y \rangle_F = \langle x, t^*y \rangle_E \quad (x \in E, y \in F)$$

(these are called the adjointable operators)

$\Rightarrow$   $t$  is  $A$ -linear and bounded (exercise)

Note: Bounded + Linear  $\not\Rightarrow$  Adjointable

$$A = C([0, i])$$

$i: F \hookrightarrow A$  bounded and linear but has no adjoint.

Facts:

- $t \in L(E, F) \Rightarrow t^* \in L(F, E)$
- $s \in L(F, G) \Rightarrow st \in L(E, G)$
- $L(E) := L(E, E)$  is a  $C^*$ -algebra with adjoint

For  $x \in E, y \in F$  define  $\Theta_{x,y} : F \rightarrow E$  by

$$\Theta_{x,y}(z) = x \underbrace{\langle y, z \rangle}_{\in A} \quad z \in F$$

$\Theta_{x,y} \in L(F, E)$  and  $(\Theta_{x,y})^* = \Theta_{y,x}$ . These are "rank 1" operators.

e.g. With  $A$  as a Hilbert  $A$ -module, (unital)

$\Theta_{1,1}$  is the identity operator.

Def:  $K(F, E)$  is the closed linear subspace of  $L(F, E)$  spanned by

$$\{\Theta_{x,y} \mid x \in E, y \in F\}$$

Note:  $K(E)$  is a closed two-sided ideal in  $L(E)$

•  $K(E)$  is not (in general) compact as operators on a Banach space.

ex:

$$K(A) \cong A$$

↑  
as a  $A$ -mod

$$\Theta_{a,b}$$

$$\longmapsto ab^*$$

↑  
dense in  
 $K(A)$

↑  
dense in  $A$

If  $A$  is unital, then  $L(A) = K(A)$ .

$$L(A) \ni t \longmapsto \Theta_{t(1), 1}$$

From  $A$ -linearity:  $t(a) = t(1a) = t(1)a$

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Riesz representation theorem for Hilbert  $A$ -module

Let  $E$  be a Hilbert  $A$ -module, fix  $x \in E$

define  $t_x: E \rightarrow A$

$$\langle a, b \rangle = a^*b$$

$$t_x y = \langle x, y \rangle$$

$t_x$  has an adjoint:

$$\langle t_x y, a \rangle_A = \langle x, y \rangle^* a = \langle y, x \rangle a = \langle y, xa \rangle$$

$$t_x^*: A \rightarrow E : t_x^*(a) = xa \Rightarrow t_x \in L(E, A)$$

$t_x$  is actually in  $K(E, A)$

- $\theta_{a, z}(y) = a \langle z, y \rangle = \langle za^*, y \rangle = t_{za^*}(y)$
- $t_x \in K(E, A)$  whenever  $x = za^*$
- use fact that  $EA = \{xa \mid x \in E, a \in A\} \subseteq E$  is dense in  $E$ .
- every element of  $K(E, A)$  is of this form (exercise).

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If  $A$  is unital  $K(E, A) = L(E, A) \dots$