C*-Algebras of Expansive Dynamical Systems

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GPOTS
May 12, 2021

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Outline

Motivation

- Smale space C^*-algebras are quite well understood (Putnam, Putnam-Spielberg, etc)
- C^*-algebras associated to expansive dynamical systems are not as well understood (K. Thomsen)
- Goal is to extend Smale space techniques in the study of expansive dynamical systems.

Dynamical Systems

A dynamical system is (X, φ) where X is a compact metric space and $\varphi : X \to X$ is a homeomorphism.

Definitions:

- (X, φ) is ***expansive*** if there exists a constant $\varepsilon_X > 0$ such that $d(\varphi^n(x), \varphi^n(y)) \leq \varepsilon_X$ for all $n \in \mathbb{Z}$ implies $x = y$.
- A point $x \in X$ is called ***non-wandering*** if for each neighborhood U of x there is an $n > 0$ such that $\varphi^n(U) \cap U \neq \emptyset$.
- (X, φ) is called ***irreducible*** if for every pair of open sets U and V there is an $n \in Z$ such that $\varphi^n(U) \cap V \neq \emptyset$.

Examples of Expansive Dynamical Systems (Shift Spaces)

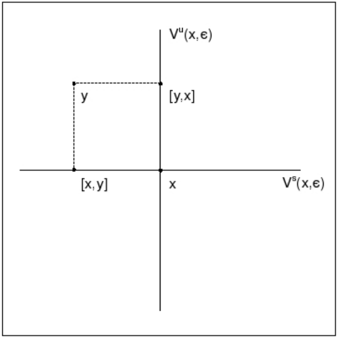
Let $\mathcal{A}$ be a finite set, consider the space $\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} \mid x_i \in \mathcal{A}\}$. The *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\sigma(x)_i = x_{i+1}$$

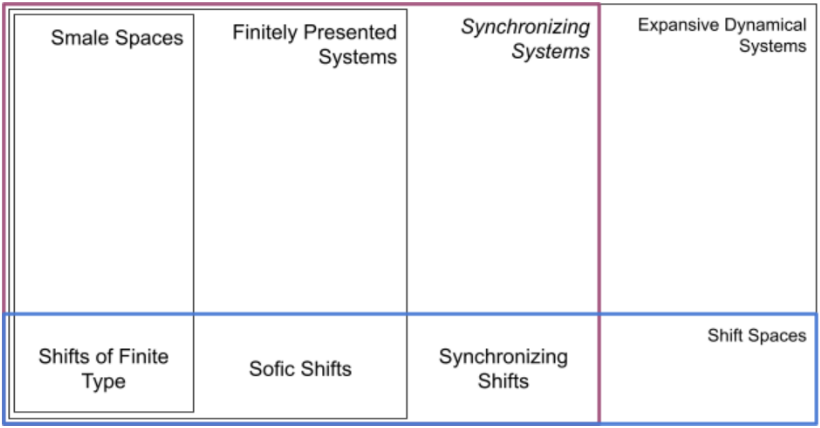
A ***shift space*** is a closed subspace $X \subseteq \mathcal{A}^{\mathbb{Z}}$ which is invariant under σ.

Shift spaces are expansive!

A ***Smale Space*** is a dynamical system that is hyperbolic in the sense that each point has a neighborhood which is homeomorphic to the product of its local expanding and contracting sets.



Map of Expansive Dynamical Systems



Following Thomsen, $x, y \in X$ are called ***locally conjugate*** if there exist open neighborhoods U and V of x and y respectively, and a homeomorphism $\gamma : U \to V$ such that $\gamma(x) = y$ and

$$\lim_{n \to \pm\infty} \sup_{z \in U} d(\varphi^n(z), \varphi^n(\gamma(z))) = 0\,.$$

This is an equivalence relation! Denote by $R \subseteq X \times X$, however we topologize R with subbase

$$\{(z, \gamma(z)) \mid z \in U\}$$

for every U, V, and γ as above.

With this topology R is an étale equivalence relation, we construct the groupoid C^*-algebra

$$A = C_r^*(R)$$

called the ***homoclinic algebra*** of (X, φ).

Remark: The ***heteroclinic algebras*** S and U are related C^*-algebras also constructed from an expansive dynamical system. For Smale spaces these are the same as the *stable* and *unstable* algebras constructed by Putnam. Note that the construction of these algebras requires periodic points to be dense!

Synchronizing Systems

Motivation: Describe a class of expansive systems with enough points that have local hyperbolic neighborhoods.

Remark: In the study of shift spaces, there is the concept of a *synchronizing word*, i.e. $w \in \mathcal{L}(X)$ such that if $uw, wv \in \mathcal{L}(X)$ then $uwv \in \mathcal{L}(X)$ where $\mathcal{L}(X)$ is the set of finite words appearing in elements X.

Definitions: Local *stable* and *unstable* sets of $x \in X$:

$$W^{\mathrm{s}}(x, \varepsilon) = \{y \in X \mid d\left(\varphi^n(x), \varphi^n(y)\right) \leq \varepsilon \text{ for all } n \geq 0\}$$
$$W^{\mathrm{u}}(x, \varepsilon) = \left\{y \in X \mid d\left(\varphi^{-n}(x), \varphi^{-n}(y)\right) \leq \varepsilon \text{ for all } n \geq 0\right\}$$

For $0 < \varepsilon \leq \dfrac{\varepsilon_X}{2}$ the intersection $W^{\mathrm{s}}(x, \varepsilon) \cap W^{\mathrm{u}}(y, \varepsilon)$ consists of at most one point (by expansiveness!).

Synchronizing Systems

For $0 < \varepsilon \leq \frac{\varepsilon_X}{2}$ define:

$$D_\varepsilon = \{(x, y) \in X \times X \mid W^{\mathrm{s}}(x, \varepsilon) \cap W^{\mathrm{u}}(y, \varepsilon) \neq \emptyset\}$$

and a map $[-, -] : D_\varepsilon \to X$ such that $[x, y] \in W^{\mathrm{s}}(x, \varepsilon) \cap W^{\mathrm{u}}(y, \varepsilon)$.

Notes:

- $[-, -]$ is continuous
- D_ε contains $\Delta_X = \{(x, x) \mid x \in X\}$ and $[x, x] = x$.

A point $x \in X$ is called ***synchronizing*** if there exists $\delta_x > 0$ such that

$$W^{\mathrm{u}}(x, \delta_x) \times W^{\mathrm{s}}(x, \delta_x) \subseteq D_\varepsilon$$

and $[-, -]$ restricted to $W^{\mathrm{u}}(x, \delta_x) \times W^{\mathrm{s}}(x, \delta_x)$ is a homeomorphism onto its image, which is a neighborhood of x.

Synchronizing Systems

An expansive dynamical system (X, φ) is called a ***sychronizing system*** if it is *irreducible* and there exists a synchronizing point $x \in X$.

Remarks:

- By irreducibility, synchronizing systems have a dense open set of synchronizing points.
- There exist expansive dynamical systems that are *not* synchronizing, e.g. *minimal* (every orbit is dense) expansive dynamical systems such as Toeplitz flows.
- *Synchronizing shifts* have been studied in symbolic dynamics.

Examples of a Synchronizing System (Even Shift)

Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be the set of all elements of $\{0,1\}^{\mathbb{Z}}$ which do not contain the word $10^{2k+1}1$ for any $k \geq 0$. This is a shift space called the ***even shift***.

The even shift is what is called a *sofic shift* (not a Smale space!).

Consider the sequence of all zeros:

$$\overline{0} = \ldots 00000 \ldots \in X$$

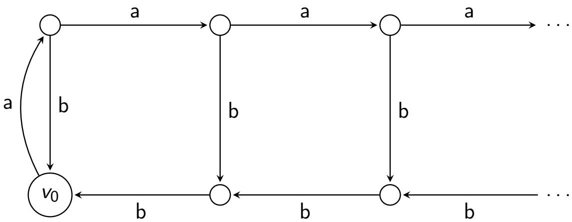
This point is *not* synchronizing!

$$\begin{aligned} x &= \ldots 111000000 \ldots \\ y &= \ldots 000000111 \ldots \\ [x,y] &= \ldots 111000111 \ldots \end{aligned}$$

Every other element of the even shift is synchronizing.

Examples of a Synchronizing System ($a^n b^n$-Shift)

Let $X \subseteq \{a, b\}^{\mathbb{Z}}$ be the closure of the set of bi-infinite paths on the following graph



This is a synchronizing system which is not finitely presented.

An element of X is synchronizing only if it is represented by a path that on the graph passing through v_0.

Results

For a dynamical system (X, φ) the set of periodic points is defined as

$$\mathrm{Per}(X, \varphi) = \{x \in X \mid \varphi^n(x) = x \text{ for some } n > 0\}$$

Theorem (Deeley, S.)

If (X, φ) is a synchronizing system then $\mathrm{Per}(X, \varphi)$ is dense in X.

Theorem (Deeley, S.)

The homoclinic algebra of an expansive dynamical system is asymptotically commutative. That is, for each $a, b \in A$

$$\lim_{n\to\infty} ||\varphi^n(a)b - b\varphi^n(a)|| = 0\,.$$

Results

We can think of the K-theory of the C^*-algebras H, S, and U as being an obstruction to (X, φ) being a Smale space, finitely presented, etc.

- For an expansive dynamical system (X, φ), if $S \otimes U$ is not Mortia equivalent to H then (X, φ) is not a Smale space.
  - For example, this is true for the even shift.
- For a shift space (X, σ), if the rank of $K_0(H)$ is not finite then X cannot be a sofic shift.
  - The $a^n b^n$-shift has infinite rank K-theory, and is not a sofic shift.

References

- K. Thomsen, *C^*-Algebras of Homoclinic and Heteroclinic Structure in Expansive Dynamical Systems*
- D. Fried, *Finitely Presented Dynamical Systems*
- I. Putnam, *C^*-Algebras From Smale Spaces*
- D. Ruelle, *Thermodynamic Formalism*

Thank you!