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Declaration

The material in Chapter 1 is expository. The material in Chapters 2, 3, 4, 5 and 6 is to the best of my knowledge original, except where otherwise indicated.

Summary

The main aim of this thesis is to investigate the relationship between the quantized enveloping algebra $U(gl_n)$ (corresponding to the Lie algebra gl_n) and the q-Schur algebra, $S_q(n, r)$. It was shown in [BLM] that there is a surjective algebra homomorphism

$$\theta_r : U(gl_n) \to \mathbb{Z}[v, v^{-1}] \otimes S_q(n, r),$$

where $q = v^2$.

§1 is devoted to background material.

In §2, we show explicitly how to embed the q-Schur algebra into the r-th tensor power of a suitable $n \times n$ matrix ring. This gives a product rule for the q-Schur algebra with similar properties to Schur's product rule for the unquantized Schur algebra. A corollary of this is that we can describe, in §2.3, a certain family of subalgebras of the q-Schur algebra.

In §3, we use the product rule of §2 to prove a q-analogue of Woodcock's straightening formula for codeterminants. This gives a basis of "standard quantized codeterminants" for $S_q(n, r)$ which is heavily used in chapters 4, 5 and 6.

In §4, we use the theory of quantized codeterminants developed in §3 to describe preimages under the homomorphism θ_r and the kernel of θ_r.

In §5, we use the results of §3 and §4 to link the representation theories of $U(gl_n)$ and $S_q(n, r)$. We also obtain a simplified proof of Dipper and James' "semistandard basis theorem" for q-Weyl modules of q-Schur algebras.

In §6, we show how to make the set of q-Schur algebras $S_q(n, r)$ (for a fixed n) into an inverse system. We prove that the resulting inverse limit, $\widehat{S}_v(n)$, is a cellular algebra which is closely related to the quantized enveloping algebra $U(sl_n)$ and Lusztig's algebra $\dot{U}$.

1. Preliminaries

In this chapter, we introduce the necessary background material for the rest of this thesis. Most of this material is expository. In §1.1, we introduce the quantized enveloping algebra, and in §1.2, we introduce the Schur algebras and q-Schur algebras. In §1.3, we state some well known properties of Coxeter systems, which we use to analyse symmetric groups and their associated Hecke algebras. Finally, §1.4 is devoted to combinatoric definitions.

The symbols q and v always represent indeterminates unless stated otherwise, and are related via $v^2 = q$. Throughout the thesis, we denote $\mathbb{Z}[v, v^{-1}]$, the ring of Laurent polynomials over $\mathbb{Z}$, by $\mathcal{A}$.

§1.1 The Quantized Enveloping Algebra

1.1.1 Definition of the Quantized Enveloping Algebra

We now define the algebra $U(gl_n)$ over $\mathbb{Q}(v)$ as in [D4]. It is generated by elements

$$E_i, F_i, K_j, K_j^{-1},$$

(where $1 \leq i \leq n-1$ and $1 \leq j \leq n$) subject to the following relations:

$$K_i K_j = K_j K_i, \tag{1}$$

$$K_i K_i^{-1} = 1, \tag{2}$$

$$K_i E_j = v^{\epsilon^+(i,j)} E_j K_i, \tag{3}$$

$$K_i F_j = v^{\epsilon^-(i,j)} F_j K_i, \tag{4}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}, \tag{5}$$

$$E_i E_j = E_j E_i \quad \text{if } |i-j| > 1, \tag{6}$$

$$F_i F_j = F_j F_i \quad \text{if } |i-j| > 1, \tag{7}$$

$$E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i-j| = 1, \tag{8}$$

$$F_j^2 F_i - (v + v^{-1}) F_j F_i F_j + F_i F_j^2 = 0 \quad \text{if } |i-j| = 1. \tag{9}$$

Here,

$$\epsilon^+(i,j) := \begin{cases} 1 & \text{if } j = i; \\ -1 & \text{if } j = i-1; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\epsilon^-(i,j) := \begin{cases} 1 & \text{if } j = i-1; \\ -1 & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

The algebra is also equipped with two coassociative comultiplications. One, $\Delta : U \to U \otimes U$, has the following effect on the generators:

$$\begin{aligned} \Delta(E_i) &= 1 \otimes E_i + E_i \otimes K_i K_{i+1}^{-1}, \\ \Delta(F_i) &= K_i^{-1} K_{i+1} \otimes F_i + F_i \otimes 1, \\ \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1}. \end{aligned}$$

The other map, $\Delta' : U \to U \otimes U$, is defined by

$$\begin{aligned} \Delta'(E_i) &= K_i K_{i+1}^{-1} \otimes E_i + E_i \otimes 1, \\ \Delta'(F_i) &= 1 \otimes F_i + F_i \otimes K_i^{-1} K_{i+1}, \\ \Delta'(K_i) &= K_i \otimes K_i, \\ \Delta'(K_i^{-1}) &= K_i^{-1} \otimes K_i^{-1}. \end{aligned}$$

We will not be using Δ' in this thesis, but we include it for completeness because many authors prefer it to Δ.

The maps Δ and Δ' are also algebra homomorphisms. This means that if M is a U-module, we can make $M^{\otimes r}$ into a U-module via

$$u.(m_1 \otimes \cdots \otimes m_r) = \Delta^{r-1}(u)(m_1 \otimes \cdots \otimes m_r).$$

This is well-defined because Δ is coassociative. Note that Δ^{r-1} is an algebra homomorphism $U \to U^{\otimes r}$. It may be checked by a simple inductive argument that the effect of Δ^{r-1} on the generators E_i, F_i, K_i, K_i^{-1} is as follows:

$$\begin{aligned} \Delta^{r-1}(E_i) =& (1 \otimes \cdots \otimes 1 \otimes E_i) + (1 \otimes \cdots \otimes 1 \otimes E_i \otimes K_i K_{i+1}^{-1}) + \cdots \\ & \cdots + (E_i \otimes K_i K_{i+1}^{-1} \otimes \cdots \otimes K_i K_{i+1}^{-1}), \\ \Delta^{r-1}(F_i) =& (K_i^{-1} K_{i+1} \otimes \cdots \otimes K_i^{-1} K_{i+1} \otimes F_i) + (K_i^{-1} K_{i+1} \otimes \cdots \otimes K_i^{-1} K_{i+1} \otimes F_i \otimes 1) + \cdots \\ & \cdots + (F_i \otimes 1 \otimes \cdots \otimes 1), \\ \Delta^{r-1}(K_i) =& K_i \otimes \cdots \otimes K_i, \\ \Delta^{r-1}(K_i^{-1}) =& K_i^{-1} \otimes \cdots \otimes K_i^{-1}. \end{aligned}$$

We introduce certain elements of $\mathbb{Q}(v)$, as follows.

We will define the *quantum integer* $[a]$, where a is a nonnegative integer, to be

$$\frac{v^a - v^{-a}}{v - v^{-1}}.$$

We also define *quantized factorials* by

$$[a]! := \prod_{k=1}^{a} [k],$$

and *quantized binomial coefficients* by

$$\begin{bmatrix} a \\ b \end{bmatrix} := \frac{[a]!}{[b]![a-b]!}.$$

Note that when v is specialised to 1, these become ordinary integers, factorials and binomial coefficients, respectively.

If X is an element of U and c is a nonnegative integer, then the *divided power* $X^{(c)}$ is defined to be

$$\frac{X^c}{[c]!}.$$

Sometimes, it is convenient to work with an *integral form* of $U(gl_n)$, which is denoted by $U_{\mathcal{A}}(gl_n)$, or (when the context is clear) by U. This is an $\mathcal{A}$-algebra which is generated by the elements of $U(gl_n)$ given by

$$E_i^{(c)} \quad (1 \leq i < n,\ c \in \mathbb{N}) \tag{10}$$

$$F_i^{(c)} \quad (1 \leq i < n,\ c \in \mathbb{N}) \tag{11}$$

$$K_j \quad (1 \leq j \leq n) \tag{12}$$

$$\begin{bmatrix} K_j; 0 \\ t \end{bmatrix} \quad (1 \leq j \leq n,\ t \in \mathbb{N}). \tag{13}$$

Here,

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} := \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

The $\mathcal{A}$-algebra $U_{\mathcal{A}}^-$ is generated by the elements in (11) subject to relations of form (7) and (9).

The $\mathcal{A}$-algebra $U_{\mathcal{A}}^0$ is generated by the elements in (12) and (13), subject to relations of form (1) and (2).

The $\mathcal{A}$-algebra $U_{\mathcal{A}}^+$ is generated by the elements in (10), subject to relations of form (6) and (8).

It is known (see [L3, §3.2]) that $U \cong U^- \otimes U^0 \otimes U^+$ as $\mathcal{A}$-modules.

A good reference for the general theory of quantized enveloping algebras is [L3].

1.1.2 Root systems and root vectors

We now state without proof some properties of root systems. The general theory of these can be found in any good text on Lie algebras.

Associated with the Lie algebra sl_n, or (in our case) gl_n, is a certain collection of vectors in $(n-1)$-dimensional Euclidean space known as a *root system* of type A_{n-1}. It is well-known that this root system contains an independent subset (the *fundamental roots*) $\{\alpha_1, \ldots, \alpha_{n-1}\}$ such that any other root is of form

$$\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \quad (1 \leq i \leq j < n)$$

or of form

$$\alpha = -\alpha_i - \alpha_{i+1} - \cdots - \alpha_j \quad (1 \leq i \leq j < n).$$

In the first case, the root α is called *positive*, and in the second case, the root α is called *negative*. Denote these two sets of roots by Φ^+ and Φ^-, respectively.

We will also write $\alpha(i, j+1)$ to denote the positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$.

We define the *height*, $h(\alpha)$, of $\alpha = \alpha(i, j)$ to be $j - i$.

Following [L2, §2.2], we define the function $g(\alpha(i, j)) = j - 1$. (The function g finds the index of the highest fundamental root occurring with nonzero coefficient in its argument.)

The bilinear map $(\ ,\ ) : \Phi^+ \times \Phi^+ \to \mathbb{Z}$ is defined to satisfy

$$(\alpha_i, \alpha_j) := \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } |i - j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Associated to each positive root α in gl_n, we define an element E_α in U^+ and an element F_α in U^-.

Let $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ be a (typical) positive root in type A_{n-1}, where the α_i are, as usual, the fundamental roots. If $i = j$, we define $E_\alpha := E_i$ and $F_\alpha := F_i$. If $i \neq j$, we let $\gamma = \alpha - \alpha_j$ and $\beta = \alpha - \alpha_i$ and define, by induction on $j - i$,

$$E_\alpha := E_\gamma E_j - v^{-1} E_j E_\gamma$$

and

$$F_\alpha := F_\beta F_i - v^{-1} F_i F_\beta.$$

We also order the elements E_α and the elements F_α as follows.

The element $E_{\alpha(i,j)}$ precedes (or appears to the left of) the element $E_{\alpha(k,l)}$ if $i > k$ or ($i = k$ and $j > l$). We denote by $\beta_N, \ldots, \beta_2, \beta_1$ the sequence of positive roots corresponding to this sequence of root vectors.

The element $F_{\alpha(i,j)}$ precedes (or appears to the left of) the element $F_{\alpha(k,l)}$ if $j < l$ or ($j = l$ and $i < k$). We denote by $\gamma_N, \dots, \gamma_2, \gamma_1$ the sequence of positive roots corresponding to this sequence of root vectors.

Example

Let $n = 4$. In this case, the positive roots are

$$\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_3.$$

The ordering on the elements E_α corresponds to the ordering

$$\alpha(3,4) < \alpha(2,4) < \alpha(2,3) < \alpha(1,4) < \alpha(1,3) < \alpha(1,2)$$

on the positive roots, and the ordering on the elements F_α corresponds to the ordering

$$\alpha(1,2) < \alpha(1,3) < \alpha(2,3) < \alpha(1,4) < \alpha(2,4) < \alpha(3,4)$$

on the positive roots.

1.1.3 Basis theorems for Quantized Enveloping Algebras

In this section, we work with the integral form $U = U_{\mathcal{A}}$. The aim is to define a Poincaré-Birkhoff-Witt type basis for U by using the elements E_α and F_α which were introduced in §1.1.2.

Lemma 1.1

Define $\psi^+ : U^+ \to U^+$ by $\psi^+(E_i) := E_{n-i}$. Then ψ^+ extends naturally to an $\mathcal{A}$-algebra isomorphism $\psi^+ : U^+ \to U^+$.

Define $\psi^- : U^- \to U^-$ by $\psi^-(F_i) := F_{n-i}$. Then ψ^- extends naturally to an $\mathcal{A}$-algebra isomorphism $\psi^- : U^- \to U^-$.

Proof

The map ψ^+ is self-inverse, and preserves the relations (6) and (8).

The map ψ^- is self-inverse, and preserves the relations (7) and (9). ∎

Lemma 1.2

There is an $\mathcal{A}$-algebra isomorphism $\omega^- : U^- \to U^+$ given by $\omega^-(F_i) = E_i$.

There is an $\mathcal{A}$-algebra isomorphism $\omega^+ : U^+ \to U^-$ given by $\omega^+(E_i) = F_i$.

Proof Since ω^- and ω^+ are mutual inverses, and they are clearly surjective, it suffices to check that each one preserves the relations. This is immediate from the nature of the relations (6), (7), (8) and (9). ∎

Lemma 1.3

(i) $\psi^+(\omega^-(F_{\alpha(i,j)})) = E_{\alpha(n+1-j,n+1-i)}$.

(ii) F_α precedes F_β in the ordering on the elements F_γ if and only if $\psi^+(\omega^-(F_\alpha))$ precedes $\psi^+(\omega^-(F_\beta))$ in the ordering on the E_γ.

Proof

We first prove (i), using induction on $h = h(\alpha)$. The case $h = 1$ follows from the definition of ω^-.

For the general case, $F_{\alpha(i,j)} = F_{\alpha(i+1,j)}F_{\alpha(i,i+1)} - v^{-1}F_{\alpha(i,i+1)}F_{\alpha(i+1,j)}$, by definition. By induction, we have

$$\psi^+(\omega^-(F_{\alpha(i,j)})) = E_{\alpha(n-j+1,n-i)}E_{\alpha(n-i,n-i+1)} - v^{-1}E_{\alpha(n-i,n-i+1)}E_{\alpha(n-i,n-j+1)},$$

because $\psi^+\omega^-$ is an algebra isomorphism. The result now follows from the defintion of

$$E_{\alpha(n-j+1,n-i+1)}.$$

The proof of (ii) is immediate from the claim of (i) and the definitions of the two orders. ∎

Definition

Define V^- to be the $\mathcal{A}$-algebra given by generators $\{\widehat{F}_\alpha^{(c)} : \alpha \in \Phi^+, c \in \mathbb{Z}_{\geq 0}\}$ (where $F_\alpha^{(0)} = 1$) and relations

$$\widehat{F}_\alpha^{(c)}\widehat{F}_\alpha^{(b)} = \begin{bmatrix} c+b \\ c \end{bmatrix} \widehat{F}_\alpha^{(c+b)}, \tag{14}$$

$$\widehat{F}_{\alpha_i}^{(c)}\widehat{F}_\alpha^{(b)} = \widehat{F}_\alpha^{(b)}\widehat{F}_{\alpha_i}^{(c)} \qquad \text{if } (\alpha, \alpha_i) = 0 \text{ and } i < g(\alpha), \tag{15}$$

$$\widehat{F}_\alpha^{(b)}\widehat{F}_{\alpha'}^{(c)} = \sum_{j \geq 0; j \leq c; j \leq b} v^{-j-(c-j)(b-j)}\widehat{F}_{\alpha'}^{(c-j)}\widehat{F}_{\alpha+\alpha'}^{(j)}\widehat{F}_\alpha^{(b-j)}, \tag{16}$$

$$v^{cb}\widehat{F}_{\alpha'}^{(c)}\widehat{F}_{\alpha+\alpha'}^{(b)} = \widehat{F}_{\alpha+\alpha'}^{(b)}\widehat{F}_{\alpha'}^{(c)}, \tag{17}$$

$$v^{cb}\widehat{F}_{\alpha+\alpha'}^{(b)}\widehat{F}_\alpha^{(c)} = \widehat{F}_\alpha^{(c)}\widehat{F}_{\alpha+\alpha'}^{(b)}. \tag{18}$$

The relations (16), (17) and (18) are each subject to the restrictions that $(\alpha, \alpha') = -1$ and either ($\alpha' = \alpha_i$ and $i < g(\alpha)$) or ($h(\alpha') = h(\alpha) + 1$ and $g(\alpha') = g(\alpha)$).

Proposition 1.4 (Lusztig)

There is an $\mathcal{A}$-algebra isomorphism $\phi : V^- \to U^-$ satisfying $\phi(\widehat{F}_{\alpha_i}) = F_i$, and the set

$$\left\{ \prod_{\alpha \in \Phi^+} \widehat{F}_\alpha^{(c_\alpha)} : c_\alpha \in \mathbb{Z}_{\geq 0} \right\}$$

is an $\mathcal{A}$-basis for V^-, where the order taken for the product is the same as the order on our elements F_α.

Proof

The required isomorphism is exhibited in [L2, Theorem 4.5]. This theorem also shows that with a certain fixed order, the products as shown above form an $\mathcal{A}$-basis for V^-. Fortunately, this fixed order (which is the reverse of the order shown in [L2, 2.9 (a)]) is exactly the same as the order we imposed on the elements F_α!

From [L2, Corollary 4.3] and [L2, Proposition 1.8 (d)], we see that $\phi(\widehat{F}_{\alpha_i}) = F_i$. This completes the proof. ∎

Proposition 1.5

(i) The set

$$B^- := \left\{ \prod_{\alpha \in \Phi^+} F_\alpha^{(c_\alpha)} : c_\alpha \in \mathbb{Z}_{\geq 0} \right\},$$

where the product is taken in the order corresponding to that on the elements F_α, is an $\mathcal{A}$-basis for U^-.

(ii) The set

$$B^+ := \left\{ \prod_{\alpha \in \Phi^+} E_\alpha^{(c_\alpha)} : c_\alpha \in \mathbb{Z}_{\geq 0} \right\},$$

where the product is taken in the order corresponding to that on the elements E_α, is an $\mathcal{A}$-basis for U^+.

Proof

The result (ii) will follow from Lemma 1.3 and (i), so it is enough to prove (i).

To prove (i), notice that the relation (16) shows that

$$v^{-1}\widehat{F}_{\alpha(i,j)} = \widehat{F}_{\alpha(i+1,j)}\widehat{F}_{\alpha(i,i+1)} - v^{-1}\widehat{F}_{\alpha(i,i+1)}\widehat{F}_{\alpha(i+1,j)}$$

Since we know that $F_{\alpha(i,i+1)} = \widehat{F}_{\alpha(i,i+1)}$, we now see by an induction on $h(\alpha)$ that $F_\alpha = v^{-h(\alpha)+1}\widehat{F}_\alpha$. Since the claim of (i) is true if we replace F by $\widehat{F}$ whenever it appears in the statement, and the element F_α differs from $\widehat{F}_\alpha$ by a unit in $\mathcal{A}$, we see that (i) holds. ∎

§1.2 Schur Algebras and q-Schur Algebras

1.2.1 The Classical Schur Algebra

Denote by Θ_r the set of $n \times n$ matrices with nonnegative integer coefficients whose entries sum to r.

The classical Schur algebra, $S(n, r)$ has a basis consisting of certain elements $\xi_{i,j}$ which are defined (as in [G1, §2.3]) as follows.

Let $I(n, r)$ be the set of all ordered r-tuples of elements from the set $\mathbf{n} := \{1, \ldots, n\}$. The symmetric group $\mathcal{S}_r$ acts on the set $I = I(n, r)$ on the right by place permutation via

$$(i_1, \ldots, i_r).\pi := (i_{\pi(1)}, \ldots, i_{\pi(r)}).$$

It also acts on the set $I \times I$ as $(i, j)\pi = (i\pi, j\pi)$. We write $i \sim j$ if i and j are in the same $\mathcal{S}_r$-orbit of I, and $(i, j) \sim (i', j')$ if (i, j) and (i', j') are in the same $\mathcal{S}_r$-orbit of $I \times I$. The subscripts i and j of the element $\xi_{i,j}$ lie in $I(n, r)$, and we identify $\xi_{i,j}$ and $\xi_{i',j'}$ if and only if $(i, j) \sim (i', j')$. The set of all $\xi_{i,j}$, as (i, j) ranges over a transversal Ω of all $\mathcal{S}_r$-orbits of $I \times I$ can be shown (see e.g. [G1]) to be a basis for $S(n, r)$. We will usually write ξ_l as shorthand for $\xi_{l,l}$.

Schur's product rule for the basis $\{\xi_{i,j}\}$ is given by

$$\xi_{i,j}\xi_{k,l} = \sum_{p,q \in \Omega} Z(i, j, k, l, p, q)\xi_{p,q},$$

where $Z(i, j, k, l, p, q) := |\{s \in I : (i, j) \sim (p, s) \text{ and } (k, l) \sim (s, q)\}|$.

In this thesis, we will often work with another form of the above basis for $S(n, r)$, this time indexed by Θ_r. A typical basis element will be denoted by e_A, where A is the matrix in question.

Let $i = (i_1, \ldots, i_r)$ and $j = (j_1, \ldots, j_r)$ be elements of I. The identification between the two forms of the basis is given by

$$\xi_{i,j} = e_A,$$

where the (x, y)-entry of the matrix A is given by the number of pairs (i_s, j_s) such that $i_s = x$ and $j_s = y$. It is easily seen that this is well-defined and that the matrix A lies in Θ_r.

1.2.2 The q-Schur Algebra

Let V be a vector space of dimension r over a field F, and let $\mathcal{F}$ be the set of all n-step flags

$$V_1 \subset V_2 \subset \cdots \subset V_n = V.$$

The group $G = GL(V)$ acts naturally on $\mathcal{F}$, hence on $X = \mathcal{F} \times \mathcal{F}$. Choose $(f, f') \in X$. Then

$$f = (V_1 \subset V_2 \subset \cdots \subset V_n), \quad \text{and } f' = (V_1' \subset V_2' \subset \cdots \subset V_n').$$

Set $V_0 = V_0' = \{0\}$ and define

$$a_{ij} = \dim(V_{i-1} + (V_i \cap V_j')) - \dim(V_{i-1} + (V_i \cap V_{j-1}')).$$

The map from (f, f') to (a_{ij}) induces a bijection between the set of G orbits on X and the set Θ_r (see [D4]). Define $\mathcal{O}_A$ to be the G-orbit corresponding to $A \in \Theta_r$.

Now suppose F as above is a finite field with q elements. It is shown in [BLM] that for $A, A', A'' \in \Theta_r$, there exists a function $g_{A,A',A'',q}$ given by

$$g_{A,A',A'',q} := |\{f \in \mathcal{F} : (f_1, f) \in \mathcal{O}_A, (f, f_2) \in \mathcal{O}_{A'}\}| = c_0 + c_1 q + \cdots + c_m q^m,$$

where the c_i are integers that do not depend on the prime power q, and $(f_1, f_2) \in \mathcal{O}_{A''}$. The $\mathbb{Z}[v^2]$-polynomial $g_{A,A',A''}$ is defined by

$$g_{A,A',A''} := c_0 + c_1 v^2 + \cdots + c_m v^{2m}.$$

We now define (following [D4] or [BLM, Proposition 1.2]) the q-Schur algebra, $S_q(n,r)$, to be the $\mathbb{Q}(v)$-vector space with basis $\{e_A : A \in \Theta_r\}$ with associative multiplication given by

$$e_A e_{A'} = \sum_{A'' \in \Theta_r} g_{A,A',A''} e_{A''}.$$

Du [D4] remarks that this algebra is canonically isomorphic to the q-Schur algebra defined by Dipper and James, by exhibiting the following correspondence between basis elements e_A and basis elements $\phi^d_{\lambda\mu}$ as defined by Dipper and James in [DJ3]. Here, the elements λ and μ lie in $\Lambda(n,r)$, which is the set of compositions of r into n parts, and $d \in \mathcal{D}_{\lambda\mu}$, which is the set of distinguished W_λ–W_μ double coset representatives for the Young subgroups W_λ and W_μ of $\mathcal{S}_r$, the symmetric group on r letters. Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. For each $\alpha \in \mathbf{n}$, we define I_α to be the subset of $\mathbf{n}$ given by

$$I_\alpha := \{\lambda_1 + \cdots + \lambda_{\alpha-1} + 1, \lambda_1 + \cdots + \lambda_{\alpha-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_\alpha\}.$$

Similarly, we define J_α to be the analogous subset of $\mathbf{n}$ corresponding to μ and α. Given a Dipper-James basis element, $\phi^d_{\lambda\mu}$, we define a corresponding matrix A via

$$A_{\alpha\beta} := |d(J_\beta) \cap I_\alpha|.$$

This procedure sets up the required isomorphism by sending $\phi^d_{\lambda\mu}$ to e_A. It should be noted that λ corresponds to the sums of the rows of A, and μ to the sums of the columns of A.

We will also be using certain elements $[A]$ in $S_v(n,r)$. These are closely related to the basis elements e_A via

$$[A] := v^{-\dim \mathcal{O}_A + \dim pr_1(\mathcal{O}_A)} e_A.$$

Here, the map pr_1 is the first projection from X to $\mathcal{F}$. Beilinson et al. [BLM, 2.3] prove that

$$\dim \mathcal{O}_A - \dim pr_1(\mathcal{O}_A) = \sum_{i,j,k,l} A_{ij} A_{kl},$$

where the indices are required to satisfy $i \geq k$ and $j < l$.

For $i, j \in I(n,r)$, we saw in §1.2.1 that $\xi_{i,j} = e_A$, where $\xi_{i,j} \in S(n,r)$. We now define the corresponding element $\xi_{i,j} \in S_q(n,r)$ as the element $e_A \in S_q(n,r)$, where A is as above. Thus the set $\{\xi_{i,j}\}$, as (i,j) ranges over a transversal Ω of orbits of $I \times I$, is a basis for $S_q(n,r)$. (We are using the notation $\xi_{i,j}$ for an element of $S_q(n,r)$ and for an element of $S(n,r)$, but it will be clear which is meant from context.)

The following well-known facts about the multiplication in $S_q(n,r)$ are important. (Proofs can be found in [DJ3, §2].)

(i) $\xi_{i,j}\xi_{k,l} = 0$ unless $j \sim k$, in which case it is nonzero.

(ii) $\xi_i \xi_{i,j} = \xi_{i,j}$.

(iii) $\xi_{i,j}\xi_j = \xi_{i,j}$.

1.2.3 The connection between q-Schur Algebras and Quantized Enveloping Algebras

One of the main aims of this thesis is to investigate the relationship between q-Schur algebras and quantized enveloping algebras.

Beilinson et al. [BLM, §5.7] define a surjective algebra homomorphism $\theta : U \to S_v(n,r)$. This makes $S_v(n,r)$ into a U-module, and shows that it is a quotient of U. This is given, following [D4], as follows:

$$\theta(E_i) = \sum_{D\in\mathbb{D}, E_{i,i+1}+D\in\Theta_r} [E_{i,i+1}+D],$$
$$\theta(F_i) = \sum_{D\in\mathbb{D}, E_{i+1,i}+D\in\Theta_r} [E_{i+1,i}+D],$$
$$\theta(K_i) = \sum_{D\in\mathbb{D}_r} v^{d_i}[D],$$
$$\theta(K_i^{-1}) = \sum_{D\in\mathbb{D}_r} v^{-d_i}[D].$$

Here, $\mathbb{D}$ is the set of diagonal matrices, and $\mathbb{D}_r$ means $\mathbb{D}\cap\Theta_r$. The matrix $E_{a,b}$ has 1 in the (a,b) position and zeros elsewhere, and the matrix D is of the form $\mathrm{diag}(d_1,\dots,d_n)$.

§1.3 Coxeter Systems and their Properties

In the course of this thesis we will be working with symmetric groups and their associated Hecke algebras. These objects arise from Coxeter systems of type A. We therefore introduce in this section the properties of general Coxeter systems.

1.3.1 Coxeter Groups and Hecke Algebras

Following [H, p. 105], we define a *Coxeter system* to be a pair (W,S) consisting of a group W and a set of generators $S\subset W$, subject only to relations of the form

$$(ss')^{m(s,s')}=1,$$

where $m(s,s)=1$ and $m(s,s')=m(s',s)\geq 2$ for $s\neq s'$ in S. When the presentation is understood, we shall refer to W as a *Coxeter group*.

We shall be particularly concerned with the Coxeter group of type A_{r-1}. This can be thought of as the symmetric group on r letters, $\mathcal{S}_r$. As a Coxeter group, it is given by generators $s_1,\dots,s_{r-1}$, where $m(s_i,s_j)=2$ if $|i-j|>1$ and $m(s_i,s_j)=3$ if $|i-j|=1$. The element s_i is identified with the simple transposition $(i,i+1)$.

Define the *length* of $w\in W$, denoted by $\ell(w)$, to be the smallest r for which an expression $w=s_{i_1}\cdots s_{i_r}$ exists. Such a minimal expression for w is called a *reduced* expression.

From any Coxeter system with a finite Coxeter group, we construct the so called *Hecke algebra*, denoted by $\mathcal{H}(W)$. This has basis elements T_w parametrised by $w\in W$, and the following multiplication rules:

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw)>\ell(w); \\ qT_{sw}+(q-1)T_w & \text{if } \ell(sw)<\ell(w). \end{cases}$$

(The symbol q appearing in this definition is an indeterminate, as explained earlier.)

The existence of this algebra structure is proved in [H, §7].

It follows from the relations that if $w = s_{i_1} \cdots s_{i_r}$ is reduced, then $T_w = T_{s_1} \cdots T_{s_r}$. It also follows that if $\ell(uw) = \ell(u) + \ell(w)$ then $T_{uw} = T_u T_w$.

When q is replaced by 1, it is known that the algebra $\mathcal{H} = \mathcal{H}(W)$ is canonically isomorphic to the group algebra of W.

We will require the following property of Hecke algebras:

Lemma 1.6 Let $d \in W$. Then

$$T_d . \left(\sum_{w \in W} T_w \right) = q^{\ell(d)} . \left(\sum_{w \in W} T_w \right) .$$

Proof This is well-known (see [DJ1, §3.2]) and follows by induction on $\ell(d)$. ∎

1.3.2 Parabolic Subgroups and Young Subgroups

Consider a Coxeter system (W, S). For any subset $I \subset S$, we can define a subgroup W_I of W, generated by the set I. This is called a *parabolic subgroup* of W. The following properties of parabolic subgroups are proved in [H, Theorem 5.5].

(i) For each subset I of S, the pair (W_I, I) with the given values $m(s, s')$ is a Coxeter system. Let ℓ_I denote the length function for this Coxeter system.

(ii) Let $I \subset S$. If $w = s_{i_1} \cdots s_{i_r}$ $(s_i \in S)$ is a reduced expression, and $w \in W_I$, then all $s_i \in I$. In particular, the function ℓ agrees with ℓ_I on W_I, and $W_I \cap S = I$.

(iii) The assignment $I \mapsto W_I$ defines a lattice isomorphism between the collection of subsets of S and the collection of subgroups W_I of W.

(iv) S is a minimal generating set for W.

Let $\lambda \in \Lambda(n, r)$. We can associate to λ a certain parabolic subgroup known as the *Young subgroup*. This is defined to be the subgroup $\mathcal{S}_\lambda$ of $\mathcal{S}_r$ which is generated by all simple transpositions $(p, p+1)$ such that p and $p+1$ are in the same segment of the composition of r given by λ.

1.3.3 The Length Function for Symmetric Groups

The length function for a general Coxeter system was defined in §1.3.1. Dipper and James [DJ1, §1] recall an alternate definition of the length function, applicable to symmetric groups, which is formulated rather differently. Suppose $w \in \mathcal{S}_r$. They define

$$\ell(w) := \{(i, j) : i < j \text{ and } w(i) > w(j)\}.$$

They also state the following useful identities:

$$\ell(s_i w) = \begin{cases} \ell(w) + 1 & \text{if } w(i) < w(i+1); \\ \ell(w) - 1 & \text{if } w(i) > w(i+1); \end{cases}$$
$$\ell(w s_i) = \begin{cases} \ell(w) + 1 & \text{if } w^{-1}(i) < w^{-1}(i+1); \\ \ell(w) - 1 & \text{if } w^{-1}(i) > w^{-1}(i+1). \end{cases}$$

1.3.4 Distinguished Coset Representatives

Let W_I be a parabolic subgroup of the Coxeter group W, corresponding to the subset $I \subset S$. We now define

$$\mathcal{D}_I := \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in I\}.$$

By using a reversed version of [H, Proposition 1.10], we find that for any $w \in W$, there exists a unique $a \in W_I$ and $b \in \mathcal{D}_I$ such that $w = ab$ and $\ell(w) = \ell(a) + \ell(b)$. The element b is the unique right coset representative of W_I of shortest length in W, and is called the *distinguished right coset representative* of w (relative to W_I).

1.3.5 Poincaré Polynomials

Associated with each finite Coxeter system (W, S) is a polynomial $W(t)$ known as the *Poincaré polynomial.* This is defined as follows.

Set $a_n := |\{w \in W : \ell(w) = n\}|$. Then

$$W(t) := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{\ell(w)}.$$

It is convenient to introduce at this point two vector spaces (known as *tensor space* and *tensor matrix space*) which play key rôles in this thesis. They will both turn out to be right modules for the Hecke algebra of $\mathcal{S}_r$, although we will not show this for tensor matrix space until later.

1.3.6 Tensor Space

Let V be the $\mathbb{Q}(v)$-vector space with basis $(e_1, \ldots, e_n)$. Then $T^r(V) = V^{\otimes r}$ can be made into a U–$\mathcal{H}$ bimodule as follows. Firstly we can make V into a left U-module via

$$E_i e_{i+1} = e_i,$$

$$E_i e_j = 0 \quad (j \neq i+1),$$

$$F_i e_i = e_{i+1},$$

$$F_i e_j = 0 \quad (j \neq i),$$

$$K_i e_i = v e_i,$$

$$K_i e_j = e_j \quad (j \neq i).$$

We now make $T^r(V)$ into a left U-module via the homomorphism Δ^{r-1}. Following [D4], we now make $T^r(V)$ into a right $\mathcal{H}$-module via

$$e_{i_1} \otimes \cdots \otimes e_{i_r} T_{p,p+1} := \begin{cases} v e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} & \text{if } i_p < i_{p+1}; \\ v^2 e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p = i_{p+1}; \\ v e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} + (v^2-1) e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p > i_{p+1}. \end{cases}$$

Jimbo [J] proved that there exists a quotient $\rho_r(U)$ of U such that $\rho_r(U) = \mathrm{End}_{\mathcal{H}}(T^r(V))$ and $\mathcal{H} = \mathrm{End}_U(T^r(V))$ when $n \geq r$. It is shown in [DJ3, Theorem 6.6] that the q-Schur algebra $S_q(n,r)$ is the centralising algebra of $\mathcal{H}$ with respect to "q-tensor space". It is shown in [D4] that Dipper and James' so-called q-tensor space is essentially the same as the classical tensor space defined above, but with certain powers of v introduced. Since the rings of $\mathcal{H}$-endomorphisms of the two spaces are isomorphic, we may work with either, and in this thesis, we will mainly be using classical tensor space. Thus, the v-Schur algebra can be identified with the quotient of U which acts faithfully on $T^r(V)$.

Sometimes it will be more convenient to write u_i for $e_{i_1} \otimes \cdots \otimes e_{i_r}$, where i is the element of $I(n,r)$ given by $(i_1, \ldots, i_r)$.

We define the function $m : I(n,r) \to \mathbb{Z}$ by

$$m(i) := |\{\langle a,b \rangle : 1 \leq a < b \leq r : i_a > i_b\}|.$$

We will write $[u_i]$ for $v^{-m(i)} u_i$.

The action of $\mathcal{H}$ can now be re-expressed as follows. Let $s = (p, p+1)$. Then

$$[u_i].T_s = \begin{cases} q[u_{i.s}] & \text{if } i_p \leq i_{p+1}; \\ [u_{i.s}] + (q-1)[u_i] & \text{if } i_p > i_{p+1}. \end{cases}$$

The action of T_s^{-1} is also easy to describe:

$$[u_i].T_s^{-1} = \begin{cases} q^{-1}[u_{i.s}] & \text{if } i_p \geq i_{p+1}; \\ [u_{i.s}] + (q^{-1}-1)[u_i] & \text{if } i_p < i_{p+1}. \end{cases}$$

1.3.7 Tensor Matrix Space

Note that $M_n = M_n(R)$, the $n \times n$ matrix algebra over R (where $R = \mathbb{Q}$ for the classical case and $R = \mathbb{Q}(v)$ for the quantized case), has a natural U-module structure as follows. Denote by e_{ij} or $e_{i,j}$ the matrix unit with 1 in the (i,j)-position and zeros elsewhere. A quick check shows that there is an algebra homomorphism θ_1 from U to M_n satisfying:

$$\begin{aligned}
\theta_1(E_i) &= e_{i,i+1},\\
\theta_1(F_i) &= e_{i+1,i},\\
\theta_1(K_i) &= \sum_{j=1}^{n} v^{\delta(i,j)} e_{j,j},\\
\theta_1(K_i^{-1}) &= \sum_{j=1}^{n} v^{-\delta(i,j)} e_{j,j}.
\end{aligned}$$

We will regard M_n as a left U-module via the action

$$u.m := \theta_1(u) \times m.$$

We define *tensor matrix space* to be $T^r(M_n)$, the r-th tensor power of M_n. This vector space has an associative multiplication defined on it given by

$$e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r} \times e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r} := \begin{cases} e_{a_1 d_1} \otimes \cdots \otimes e_{a_r d_r} & \text{if } b_i = c_i \text{ for all } i; \\ 0 & \text{otherwise.} \end{cases}$$

We regard $T^r(M_n)$ as a U-module via the comultiplication Δ.

Sometimes it will be more convenient to write $u_{i,j}$ for

$$e_{i_1,j_1} \otimes \cdots \otimes e_{i_r,j_r},$$

where $i, j \in I(n,r)$ are given by $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_r)$ respectively.

The symmetric group $\mathcal{S}_r$ acts on $T^r(M_n)$ on the right by place permutation of the tensors, which corresponds, in the $u_{i,j}$ notation, to the action

$$u_{i,j}.\pi = u_{i.\pi, j.\pi},$$

where the action of $\mathcal{S}_r$ on $I(n,r)$ is as in §1.2.1.

We say that a typical basis element $u_{i,j}$ of $T^r(M_n)$ is is of *parabolic form* if the stabilizer of j under the action of $\mathcal{S}_r$ on $I(n,r)$ is a parabolic subgroup. We will also say in this case that the element j is of parabolic form. An equivalent way of looking at this is that if $1 \leq a < b < c \leq r$ and $j_a = j_c$, then $j_a = j_b = j_c$.

§1.4 Combinatorics

In this section we introduce some combinatoric definitions which will be important for later purposes.

1.4.1 Two Orders

Let A and B be ordered sets. Then the *Hebrew lexicographic order* on $A \times B$ is given by $\langle a, b\rangle < \langle c, d\rangle$ if and only if $b < d$ or ($b = d$ and $a < c$). The elements of a sequence $(i_1, \ldots, i_m)$ in $A \times B$ are said to appear in Hebrew lexicographic order if $i_1 \leq \cdots \leq i_r$ with respect to this order.

A basis element $u_{i,j}$ of $T^r(M_n)$ is said to be *H-relaxed* if the elements of the sequence

$$(\langle i_1, j_1\rangle, \ldots, \langle i_r, j_r\rangle)$$

appear in Hebrew-lexicographic order.

Let A and B be ordered sets. Then the *relaxed order* on $A \times B$ is given by $\langle a, b\rangle < \langle c, d\rangle$ if and only if $b < d$ or ($b = d$ and $a > c$). The elements of a sequence $(i_1, \ldots, i_m)$ in $A \times B$ are said to appear in relaxed order if $i_1 \leq \cdots \leq i_r$ with respect to this order.

A basis element $u_{i,j}$ of $T^r(M_n)$ is said to be *relaxed* if the elements of the sequence

$$(\langle i_1, j_1\rangle, \ldots, \langle i_r, j_r\rangle)$$

appear in the relaxed order.

Examples

Let $j = (1, 1, 1, 1, 3, 3, 3, 7, 7, 7)$.

If $i = (1, 4, 6, 7, 2, 4, 5, 3, 3, 4)$ then $u_{i,j}$ is H-relaxed.

If $i' = (7, 6, 4, 1, 5, 4, 2, 4, 3, 3)$ then $u_{i',j}$ is relaxed.

1.4.2 Tableaux and Codeterminants

We now make some combinatorial definitions, most of which appear in [G2].

We define a *codeterminant* to be a nonzero product of two basis elements $\xi_{a,b}\xi_{c,d}$ of $S(n, r)$. (The codeterminant is determined by the given factorisation, and not by the element of $S(n, r)$ which is equal to it.)

We identify $\Lambda(n, r)$, the set of compositions of r into at most n pieces, with the set

$$\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_\nu \in \mathbb{N}_0 \text{ for all } \nu \in \mathbf{n} \text{ and } \sum_{\nu=1}^{n} \lambda_\nu = r\}.$$

We also define

$$\Lambda^+ = \Lambda^+(n,r) := \{\lambda \in \Lambda(n,r) : \lambda_1 \geq \cdots \geq \lambda_n\}.$$

(Here, $\mathbb{N}_0$ is the set of natural numbers, including 0.)

An element of Λ is called a *weight*, and is *dominant* if $\lambda \in \Lambda^+$. There is an obvious correspondence between elements of Λ^+ and partitions of r into not more than n parts.

The lexicographic order on Λ is defined as follows. $\mu \succ \lambda$ if for some c, $\mu_a = \lambda_a$ for all $a < c$, and $\mu_c > \lambda_c$.

The weight $\mathrm{wt}(i)$ of an element $i \in I(n,r)$ is the element $\alpha \in \Lambda$ given by $\alpha_\nu = |\{\rho \in \mathbf{r} : i_\rho = \nu\}|$ for all $\nu \in \mathbf{n}$. If $i, j \in I$, it is clear that $i \sim j$ if and only if $\mathrm{wt}(i) = \mathrm{wt}(j)$.

For each $\lambda \in \Lambda^+$ we define a *basic λ-tableau* T^λ by writing the integers $1, \ldots, r$ into a Young diagram in some arbitrary (but henceforth fixed) order. (In practice, the order we pick will always be row by row, starting with the top row, and filling each row from left to right.) To each $i \in I$ we now associate the λ-tableau $T_i^\lambda = iT^\lambda$. For example, let $n = 4$, $r = 7$, $\lambda = (4, 2, 1, 0)$. Using our choice of basic λ-tableau, then

$$T^\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array},$$

and

$$T_i^\lambda = \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & i_3 & i_4 \\ \hline i_5 & i_6 & \multicolumn{2}{c}{} \\ \cline{1-2} i_7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

If $\lambda \in \Lambda^+$ and $i \in I$, the λ-tableau T_i^λ is said to be *standard* if the elements in each row increase weakly from left to right, and the elements in each column increase strictly from top to bottom. We define

$$I_\lambda := I_\lambda(n,r) = \{i \in I : T_i^\lambda \text{ is standard}\}.$$

We say a λ-tableau T_i^λ is *row-semistandard* if the elements in each row increase weakly from left to right, and define the set

$$I'_\lambda := I'_\lambda(n,r) = \{i \in I : T_i^\lambda \text{ is row} - \text{semistandard}\}.$$

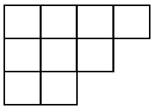
It is clear that in a standard tableau, all the entries equal to s must appear in the first s rows, by an easy induction on s. There is exactly one element of I, denoted by $\ell = \ell(\lambda)$, for which T_ℓ^λ is standard and $\mathrm{wt}(\ell) = \lambda$. (The entries in the s-th row of the tableau T_ℓ^λ are all equal to s.)

The elements of I can be lexicographically ordered as follows: $i \succeq j$ if and only if $i = j$ or, if ρ is minimal subject to the condition that $i_\rho \neq j_\rho$, then $i_\rho < j_\rho$.

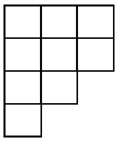
If $\lambda \in \Lambda^+$, we define the element $\lambda' \in \Lambda^+$ by the rule that λ'_h is the number of elements in column number h of the tableau of shape λ.

Example

Suppose $n = 4$, $r = 9$ and $\lambda = (4, 3, 2, 0)$. Then a tableau of shape λ is of the form



In this case, the dual tableau λ' is given by $(3, 3, 2, 1)$ and corresponds to the array given by



It is clear for any $p \in I$ that $p \sim \ell(\mathrm{wt}(p))$. Thus, any basis element $\xi_{a,b}$ of $S(n,r)$ can be written as $\xi_{i,\ell(\mu)}$ where $\mu = \mathrm{wt}(b)$, or as $\xi_{\ell(\nu),j}$ where $\nu = \mathrm{wt}(a)$. The product $\xi_{a,b}\xi_{c,d}$ will be zero unless $b \sim c$, and hence $\mathrm{wt}(b) = \mathrm{wt}(c)$, so it is easy to see that a general codeterminant can be written as

$$Y^\lambda_{i,j} := \xi_{i,\ell(\lambda)}\xi_{\ell(\lambda),j},$$

where $i, j \in I$ and $\lambda \in \Lambda$. We say that $Y^\lambda_{i,j}$ is a codeterminant *of shape* λ. If, in addition, $\lambda \in \Lambda^+$ and i and j lie in I_λ, we call the codeterminant $Y^\lambda_{i,j}$ a *standard* codeterminant. It should be noted that if $b \sim c$ then $\xi_{a,b}\xi_{c,d}$ is nonzero. This is a corollary of Schur's product rule.

We define a *q-codeterminant* to be a product $e_A e_{A'} \in S_q(n,r)$ such that the corresponding product $e_A e_{A'}$ of $S(n,r)$ is a codeterminant. (The q-codeterminant is determined by the matrices A and A', and not solely by the element of $S_q(n,r)$ which is equal to it.) We also define a *v-codeterminant* to be an element $[A][A']$ of $S_v(n,r)$, where A and A' are as above. Standard q-codeterminants and standard v-codeterminants are defined in the obvious way. It is not hard to see that when q and v are replaced by 1, the q-codeterminants and v-codeterminants specialise to the classical codeterminants introduced above.

It is convenient to introduce the following elements of $\mathbb{Z}[q]$, as in [DJ1].

a) For an nonnegative integer n, $[n]_q := 1 + q + \cdots + q^{n-1}$.

b) Define $[n]_q! := [1]_q[2]_q \cdots [n]_q$.

Let $a, b \in I(n,r)$. Then the element $P_{a,b} \in \mathbb{Z}[q]$ is defined to be

$$\prod_{i,j=1}^{n} [c_{ij}]_q!,$$

where c_{ij} is the multiplicity of $\langle i,j\rangle$ in the family $\{\langle a_1,b_1\rangle,\ldots,\langle a_r,b_r\rangle\}$. If $t = u_{a,b}$ then $P_t := P_{a,b}$.

In [DJ3, §3], the element $w_\lambda \in \mathcal{S}_r$ is introduced. It is defined as the distinguished $\mathcal{S}_\lambda$–$\mathcal{S}_{\lambda'}$ double coset representative with the property that

$$w_\lambda^{-1}\mathcal{S}_\lambda w_\lambda \cap \mathcal{S}_{\lambda'} = 1.$$

The important point to notice is that when the element of $I(n,r)$ corresponding to $T^\lambda.w_\lambda$ is written into a tableau of shape λ' in the usual way, then it is the transpose of T^λ. For example, if T^λ is given by

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}$$

then, as a tableau of shape λ', the element $(1,2,3,4,5,6,7,8,9).w_\lambda$ is given by

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 6 & 9 \\ \hline 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}.$$

1.4.3 Dominance Order

We define the partial order of *dominance*, $\trianglelefteq$, on $\Lambda^+(n,r)$ by stipulating that $\lambda \triangleleft \mu$ (μ dominates λ) if

$$\lambda_1 + \cdots + \lambda_s \leq \mu_1 + \cdots + \mu_s$$

for all $1 \leq s \leq n$.

Following [M, §3.1], we call $\pi \subseteq \Lambda^+(n,r)$ an *increasing saturation* if for any $\lambda \in \pi$ and $\mu \triangleright \lambda$ we have $\mu \in \pi$.

We define the total order $\succ$ on $\Lambda(n,r)$ by $\mu \succeq \lambda$ if $\mu = \lambda$ or the smallest i such that $\mu_i \neq \lambda_i$ satisfies $\mu_i > \lambda_i$. It is not hard to see that $\succeq$ refines $\trianglerighteq$, i.e. that $\mu \trianglerighteq \lambda$ implies $\mu \succeq \lambda$.

We label the elements of Λ^+ $\lambda^1,\ldots,\lambda^t$ where the order is chosen to satisfy the condition $\lambda^1 \succ \cdots \succ \lambda^t$.

2. A New Setting for the q-Schur Algebra

The main aim of this chapter is to describe an embedding (of rings and of U-modules) of the q-Schur algebra into the r-th tensor power $T^r(M_n)$ of the $n \times n$ matrix ring. We achieve this in Theorem 2.2.7. This embedding allows products in the q-Schur algebra to be computed in a straightforward manner, and gives a method for generalising results on $S(n, r)$ to $S_q(n, r)$. In particular we shall make use of this embedding in §3 to prove a straightening formula in $S_q(n, r)$ which generalises the straightening formula for codeterminants due to Woodcock [W].

The chapter falls into three parts. §2.1 discusses the classical Schur algebra from a new perspective compatible with the structure with the corresponding universal enveloping algebra, $\mathcal{U}(gl_n)$. §2.2 is devoted to quantum analogues of the results in §2.1. §2.3 uses the techniques of §2.2 to describe a family of subalgebras of the q-Schur algebra.

2.1. The classical Schur algebra as a subspace of tensor matrix space

We now show how to embed the classical Schur algebra in $T^r(M_n)$.

With each matrix $A \in \Theta_r$, we can (uniquely) associate a standard basis element t_A of $T^r(M_n)$, given by

$$e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

where the ordered pairs $\langle a_i, b_i \rangle$ appear in Hebrew lexicographic order, and the multiplicity with which e_{ab} occurs in t_A is the entry in the (a, b)-position of A. Recall from §1.3.7 that the symmetric group $\mathcal{S}_r$ acts on the set of basis elements of $T^r(M_n)$ on the right. Consider the orbit containing the basis element t_A, and let $\mathcal{D}_A$ be the set of distinguished right coset representatives relative to the stabilizer of t_A (which is a parabolic subgroup of $W(A_{r-1})$).

Example Let A be the matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then t_A is given by

$$e_{11} \otimes e_{11} \otimes e_{11} \otimes e_{21} \otimes e_{12} \otimes e_{22} \otimes e_{22}.$$

Definition We now define a map $\psi : S(n,r) \to T^r(M_n)$, given by

$$\psi(e_A) := t_A.\left(\sum_{w\in\mathcal{D}_A} w\right),$$

and extended by linearity.

It is important to notice that if A ranges over all elements of Θ_r, then $\psi(e_A)$ ranges over a basis for the subspace of $T^r(M_n)$ consisting of all elements t such that $t.\pi = t$ for all $\pi \in \mathcal{S}_r$. For example, if $n = 2$ and $r = 3$, and A is the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$\psi(e_A) = e_{11} \otimes e_{11} \otimes e_{12} + e_{11} \otimes e_{12} \otimes e_{11} + e_{12} \otimes e_{11} \otimes e_{11}.$$

From this, we see that dim (im ψ) is equal to the number of monomials of degree r in n^2 commuting variables, i.e. $\binom{n^2+r-1}{r}$.

The following results establish that ψ is a monomorphism of algebras, and of $\mathcal{U}$-modules, where $\mathcal{U}$ is the universal enveloping algebra of gl_n.

Proposition 2.1.1 The map $\psi : S(n,r) \to T^r(M_n)$ is a monomorphism of $\mathcal{U}$-modules.

Proof The preceding discussion of the properties of ψ shows that ψ is injective. It is known that the dimension of $S(n,r)$ is given by

$$\binom{n^2+r-1}{r},$$

which is the same as the number of monomials of degree r in n^2 commuting variables, and hence is equal to dim (im ψ). It suffices to check that the effect of E_i, F_i and H'_i is as expected, because these generate $\mathcal{U}$ as an algebra. (Here, H'_i denotes the specialisation of

$$\frac{K_i - K_i^{-1}}{v - v^{-1}}$$

in quantum gl_n. The element $H_i \in sl_n$ is related to it via $H_i = H'_i - H'_{i+1}$.) We tackle first the case of E_h. From the definition of $\theta(E_h)$ and the specialisation of [BLM, Lemma 3.4 (a2)], we see that

$$E_h.e_A = \sum_{p:a_{h+1,p}\geq 1} (a_{h,p}+1)\ e_{(A+e_{h,p}-e_{h+1,p})}.$$

Now consider the case of $E_h.\psi(e_A)$. Considering the number of different terms in $\psi(e_A)$ that give rise, under the action of E_h, to a particular term of $\psi(e_B)$ (for various matrices B of the correct type), we find that the same formula as above still holds.

The situation for F_h is entirely analogous, but uses [BLM, Lemma 3.4 (b2)].

It follows from the definition of $\theta(K_i)$ that

$$\frac{K_h - K_h^{-1}}{v - v^{-1}}.e_A = \left[\sum_{j=1}^{n} A_{hj} \right] e_A,$$

thus

$$H'_h.e_A = \sum_{j=1}^{n} A_{hj} \ e_A.$$

It is not hard to see that H'_h acts on $\psi(e_A)$ as exactly the same scalar multiple.

These observations suffice to prove the Proposition. ∎

Before showing that ψ is an isomorphism of algebras, we remark that our "tensor matrix space" is canonically isomorphic to an algebra with basis elements $u_{i,j}$ (as in §1.3.7) which appears in the literature (see [G3, §2]). The multiplication rule for the elements $u_{i,j}$ is as follows:

$$u_{i,j} u_{k,l} = \delta_{j,k} u_{i,l},$$

where δ is the Kronecker delta. The symmetric group on r letters acts on $u_{i,j}$ via $u_{i,j}^{\pi} = u_{i\pi,j\pi}$.

There is a linear map from the algebra generated by the $u_{i,j}$ into $T^r(M_n)$ determined by

$$\sigma(u_{i,j}) = e_{i_1 j_1} \otimes \cdots \otimes e_{i_r j_r}.$$

It is immediate that σ is an algebra isomorphism.

Proposition 2.1.2 $\psi : S(n,r) \to T^r(M_n)$ is an algebra monomorphism.

Proof We know ψ is injective, so we need to prove it is a ring homomorphism.

Consider a basis element $\xi_{i,j}$ of $S(n,r)$, and let $A \in \Theta_r$ be given by $\xi_{i,j} = e_A$ as in §1.2.1. We may assume the pair $\langle i, j \rangle$ is H-relaxed. We claim that

$$\psi(\xi_{i,j}) = \sum_{\pi \in X} u_{i.\pi,j.\pi},$$

where X denotes any transversal of the cosets of the stabiliser of $u_{i,j}$ in $\mathcal{S}_r$. This follows by definition of ψ in the case where X is the transversal $\mathcal{D}_A$ corresponding to $t_A = u_{i,j}$. One can see from the properties of symmetric group actions that the sum

$$\sum_{\pi \in X} u_{i.\pi,j.\pi}$$

is independent of the transversal X.

It now follows from elementary combinatoric considerations that ψ is a homomorphism, because it respects Schur's product rule. (A similar observation has been made in [G3, equation (2.3)].) ∎

Remark Note that this expression for the $\xi_{i,j}$ in terms of the $u_{i,j}$ motivates the definition of Schur's product rule.

The maps Δ and θ have classical counterparts obtained by specialising v and K_i (for all i) to be 1. Since no confusion is likely to arise, we use the symbols Δ and θ for both classical and quantized maps.

The preceding results can also be viewed in the context of the map $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$. We have seen that $S(n,1)$ is isomorphic as a ring and as a $\mathcal{U}$-module to M_n, and the same holds for the quantized version $S_v(n,1)$. Denote by θ_1 the map from $\mathcal{U}$ to M_n induced by $\theta : \mathcal{U} \to S(n,1)$. This leads to the following result.

Proposition 2.1.3 The map $\psi\theta$ agrees with the map

$$\gamma_r := \underbrace{(\theta_1 \otimes \cdots \otimes \theta_1)}_{r \text{ times}} \Delta^{r-1}$$

on $\mathcal{U}$.

Proof Firstly, observe that both maps are from $\mathcal{U}$ to $T^r(M_n)$. Both maps are algebra homomorphisms, because Δ and θ are known to be, θ_1 is a special case of θ, and ψ has been shown to be. If we can prove that the images under E_i, F_i, and H_i' are the same under both maps, the Theorem will follow.

We find that

$$\gamma_r(E_i) = \sum_{j=1}^{r} 1 \otimes \cdots \otimes 1 \otimes e_{ii+1} \otimes 1 \otimes \cdots \otimes 1,$$

where the e_{ii+1} occurs in the j-th place in the tensor. However, because

$$\sum_{i=1}^{n} e_{ii} = 1$$

in M_n, we see, by expanding the expression for γ_r, that it agrees with that given by $\psi(\theta(E_i))$.

The case of $\gamma_r(F_i)$ is entirely analogous.

In the case of H_i',

$$\gamma_r(H_i') = \sum_{j=1}^{r} 1 \otimes \cdots \otimes 1 \otimes e_{ii} \otimes 1 \otimes \cdots \otimes 1,$$

where the e_{ii} occurs at the j-th place. Expanding 1 as before, we find that each tensor occurs d_i times, where d_i is the (i,i)-entry of the corresponding matrix in $S(n,r)$. This is exactly as is required. ∎

Remark We shall see that the above result quantizes, in the sense that something similar works when Δ and θ as above are replaced by their quantum analogues. This yields a quantized version of γ_r which, again, is an algebra homomorphism. Because tensor space is made into a U-module via a process equivalent to applying γ_r, and because the v-Schur algebra can be described as the faithful quotient of U corresponding to this action, it can be seen that

$$\gamma_r(U) \cong S_v(n,r),$$

as an algebra and as a U-module. This isomorphism is not as obvious as in the classical case and it is harder to work out the images of basis elements under γ_r, but, we shall see in §2.2 that it can be done, by finding a suitable quantum analogue of ψ. This gives our product rule for elements of the q-Schur algebra.

2.2. The v-Schur algebra as a subspace of tensor matrix space

We have seen in the above sections that the classical Schur algebra $S(n,r)$ embeds in $T^r(M_n)$ in a natural way compatible with the $\mathcal{U}$-module and algebra structures of both objects. Moreover, the portion of $T^r(M_n)$ identified with the Schur algebra is invariant under the action of the symmetric group $\mathcal{S}_r$ acting on the right of $T^r(M_n)$ by place permutation. Next, we define an action of the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ (corresponding to the symmetric group on r letters) on $T^r(M_n)$ on the right. The invariant subspace of $T^r(M_n)$ corresponding to the "ind" representation of the Hecke algebra (which corresponds to the trivial representation of the symmetric group) will remarkably be seen to have a U-module structure and algebra structure which agree precisely with those of the v-Schur algebra. By specialising the parameters to 1, we recover the classical objects of the previous part of this chapter.

Definition Let

$$e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}$$

be a standard basis element of $T^r(M_n)$. We define the corresponding quantized basis element to be

$$[e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}] := v^{-m} e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

where

$$m := |\{\langle i,j\rangle : 1 \leq i < j \leq r,\ a_i > a_j\}| + |\{\langle i,j\rangle : 1 \leq i < j \leq r,\ b_i > b_j\}|.$$

If $u_{i,j}$ corresponds to $e_{a_1b_1} \otimes \cdots \otimes e_{a_rb_r}$ in the way described in §2.1, then we write $[u_{i,j}]$ for the quantity $v^* u_{i,j}$ corresponding to $[e_{a_1b_1} \otimes \cdots \otimes e_{a_rb_r}]$. (We use the convention that the symbol $*$ represents some integer.)

Definition The action of $\mathcal{H}(A_1)$ on $T^2(M_n)$ is defined as follows. Let $[e_{ab} \otimes e_{cd}]$ be a typical (quantized) basis element. Then $[e_{ab} \otimes e_{cd}].T_s$ (s being the standard generator of the Coxeter group of type A_1) is given by:

$$[e_{ab} \otimes e_{cd}].T_s := \begin{cases} q[e_{cd} \otimes e_{ab}] & \text{if } a \leq c, b \leq d; \\ [e_{cd} \otimes e_{ab}] + (q-1)[e_{ab} \otimes e_{cd}] & \text{if } a > c, b > d; \\ [e_{cd} \otimes e_{ab}] + (q-1)[e_{ad} \otimes e_{cb}] & \text{if } a > c, b \leq d; \\ q[e_{cd} \otimes e_{ab}] + (q-1)[e_{ab} \otimes e_{cd}] - (q-1)[e_{ad} \otimes e_{cb}] & \text{if } a \leq c, b > d. \end{cases}$$

As usual, we identify q with v^2. It remains to be seen that this action is well-defined.

The action of $\mathcal{H}(A_{r-1})$ on the quantized basis elements of $T^r(M_n)$ is defined entirely analogously, with a typical generator $T_{(p,p+1)}$ acting as in the $r = 2$ case, but affecting only the p-th and $(p+1)$-th positions in the tensor.

Lemma 2.2.1 The above action is a well-defined action of $\mathcal{H}(A_{r-1})$ on $T^r(M_n)$.

Proof It is well-known (see for example [H]) that $\mathcal{H}$ is given by generators $T_1, \ldots, T_{r-1}$ subject to the following relations:

$$T_s^2 = qT_e + (q-1)T_s,$$

$$T_sT_t = T_tT_s \quad \text{if } |s-t| > 1,$$

$$T_sT_tT_s = T_tT_sT_t \quad \text{if } |s-t| = 1.$$

Here, T_e is the multiplicative identity in $\mathcal{H}$, and s and t are integers between 1 and $r-1$ inclusive.

To check the first of these relations, it is enough to consider the case $r = 2$. The element T_s acts on the space spanned by the ordered basis

$$[e_{ab} \otimes e_{cd}], [e_{cd} \otimes e_{ab}], [e_{cb} \otimes e_{ad}], [e_{ad} \otimes e_{cb}],$$

where we assume that $a < c$ and $b < d$. With respect to this basis, the right action of T_s is given by the matrix

$$M := \begin{pmatrix} 0 & q & 0 & 0 \\ 1 & q-1 & 0 & 0 \\ 0 & q-1 & 0 & 1 \\ -(q-1) & 0 & q & q-1 \end{pmatrix}.$$

It is routine to check that $M^2 = qI + (q-1)M$, where I is the identity matrix. There are also three degenerate cases to check.

1. The case $a = c, b < d$. Consider the basis $[e_{ab} \otimes e_{ad}], [e_{ad} \otimes e_{ab}]$. Using the previous method, we find the action of T_s is given by

$$M := \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}.$$

   This matrix satisfies $M^2 = qI + (q-1)M$ as required.

2. The case $a < c, b = d$. This is very similar to case 1., except that the basis to take is

$$[e_{ab} \otimes e_{cb}], [e_{cb} \otimes e_{ab}].$$

3. The case $a = c, b = d$. Here, T_s acts on $[e_{ab} \otimes e_{ab}]$ via the ind representation, i.e. $T_s \mapsto q$. This satisfies the required relation.

Since $T^r(M_n)$ is spanned by quantized basis elements, each of which falls into exactly one of the above four classes, we have shown that the first relation, $T_s^2 = qT_e + (q-1)T_s$, always holds.

The second relation (known as a short braid relation), $T_s T_t = T_t T_s$ when $|s - t| > 1$, is an easy case. The action of $T_{(p,p+1)}$ is a sum of certain permutations of coordinates in the matrix units occuring in the p-th and $(p+1)$-th places, so it is clear that if $|s-t| > 1$, the permutations will not interfere with each other, and hence the elements T_s and T_t will commute with each other.

The last relation (known as a long braid relation) is the trickiest to check. To help in the checking, we recall the action of $\mathcal{H}$ on tensor space, as defined by Du [D4] and others.

Let V be, as usual, a vector space over $\mathbb{Q}(v)$ with basis $e_1, \dots, e_n$. We now define quantized basis elements of $T^r(V)$ from the usual basis as follows:

$$[e_{i_1} \otimes \cdots \otimes e_{i_r}] := v^{-m} e_{i_1} \otimes \cdots \otimes e_{i_r},$$

where

$$m := |\{\langle a, b\rangle : 1 \leq a < b \leq r,\ i_a > i_b\}|.$$

Following [D4], we see that an action of $\mathcal{H}$ on $T^r(V)$ can be defined as follows:

$$[e_{i_1} \otimes \cdots \otimes e_{i_r}].T_{p,p+1} = \begin{cases} q[e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p \leq i_{p+1}; \\ [e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] + (q-1)[e_{i_1} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p > i_{p+1}. \end{cases}$$

We will exploit the fact that this action resembles the case $b \leq d$ in our candidate for the action of $\mathcal{H}$ on $T^r(M_n)$.

To deal with the case $b > d$, we calculate the action of T_s^{-1} in each case. This follows easily from the fact that $T_s^{-1} = (q^{-1} - 1)T_e + q^{-1}T_s$ and the original definitions, because T_e acts as the identity. The action of T_s^{-1} on $T^2(M_n)$ is as follows:

$$[e_{ab} \otimes e_{cd}].T_s^{-1} := \begin{cases} q^{-1}[e_{cd} \otimes e_{ab}] & \text{if } a \geq c, b \geq d; \\ [e_{cd} \otimes e_{ab}] + (q^{-1} - 1)[e_{ab} \otimes e_{cd}] & \text{if } a < c, b < d; \\ [e_{cd} \otimes e_{ab}] + (q^{-1} - 1)[e_{ad} \otimes e_{cb}] & \text{if } a < c, b \geq d; \\ q^{-1}[e_{cd} \otimes e_{ab}] + (q^{-1} - 1)[e_{ab} \otimes e_{cd}] - (q^{-1} - 1)[e_{ad} \otimes e_{cb}] & \text{if } a \geq c, b < d. \end{cases}$$

It follows that the action of $T_{(p,p+1)}^{-1}$ on $T^r(V)$ is given by setting $[e_{i_1} \otimes \cdots \otimes e_{i_r}].T_{p,p+1}^{-1}$ equal to

$$\begin{cases} q^{-1}[e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p \geq i_{p+1}; \\ [e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] + (q^{-1} - 1)[e_{i_1} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p < i_{p+1}. \end{cases}$$

Note the similarity between the action on tensor space and the $b \geq d$ case in the prospective matrix action here. Also note that all the actions have a "continuity" property: for example in any case where $a = c$, it does not matter whether this is considered as $a \leq c$ or $a \geq c$.

Comparing the actions of T_s on $[e_{ab} \otimes e_{cd}]$ and on $[e_a \otimes e_c]$, we find that they are very similar, provided that $b \leq d$. What happens is that the second coordinates in the matrix units are exchanged, and the action on the first coordinates is exactly the same as in the action on tensor space. On the other hand, the action of T_s^{-1} on $[e_{ab} \otimes e_{cd}]$ is similar to the action of T_s^{-1} on $[e_a \otimes e_c]$, provided that $b \geq d$. Again, the second coordinates are exchanged, and the first coordinates are acted upon as in the action on tensor space.

With these observations out of the way, we proceed with the proof. Clearly, it is enough to study the case $r = 3$. Pick a quantized standard basis element, $[e_{ab} \otimes e_{cd} \otimes e_{ef}]$. It is enough to show that $T_sT_tT_s$ acts in the same way as $T_tT_sT_t$ on this basis element, where s corresponds to the permutation $(1, 2)$ and t corresponds to the permutation $(2, 3)$. The above discussion shows that this is guaranteed, provided $b \leq d \leq f$, because we know that the action on tensor space is well-defined. For the other cases, we need another method. For example, if $b > d > f$, we know from the discussion that $T_s^{-1}T_t^{-1}T_s^{-1}$ acts in the same way as $T_t^{-1}T_s^{-1}T_t^{-1}$, and this is equivalent to showing that the long braid relation holds. There are six possible cases. Here is a table showing the correct method of proof in each case. (There is some overlap—in these cases, any of the given methods will work.) For clarity, σ means T_s and τ means T_t.

Situation	Method of Proof
$b \leq d \leq f$	$\sigma\tau\sigma = \tau\sigma\tau$
$f \leq d \leq b$	$\sigma^{-1}\tau^{-1}\sigma^{-1} = \tau^{-1}\sigma^{-1}\tau^{-1}$
$d \leq b \leq f$	$\sigma^{-1}\tau\sigma = \tau\sigma\tau^{-1}$
$d \leq f \leq b$	$\sigma^{-1}\tau^{-1}\sigma = \tau\sigma^{-1}\tau^{-1}$
$b \leq f \leq d$	$\sigma\tau\sigma^{-1} = \tau^{-1}\sigma\tau$
$f \leq b \leq d$	$\sigma\tau^{-1}\sigma^{-1} = \tau^{-1}\sigma^{-1}\tau$

This suffices to prove that the long braid relation is preserved.

This completes the proof. ∎

It will be important in the proofs which follow that the actions of $U(gl_n)$ and $\mathcal{H}(A_{r-1})$ on $T^r(M_n)$ commute with each other. The action of U on M_n is given by:

$$E_i e_{jk} := \delta_{i,j-1} e_{j-1k},$$

$$F_i e_{jk} := \delta_{i,j} e_{j+1k},$$

$$K_i e_{jk} := v^{\delta_{i,j}} e_{jk},$$

$$K_i e_{jk} := v^{-\delta_{i,j}} e_{jk},$$

where δ is the Kronecker delta. The action extends to $T^r(M_n)$ via Δ^{r-1}.

Lemma 2.2.2 The actions of $\mathcal{H}(A_{r-1})$ and U on $T^r(M_n)$ commute with each other.

Proof We will rely heavily on the well-known fact that the actions of U and $\mathcal{H}$ on tensor space commute with each other. Note that our action of U on $T^r(M_n)$ is essentially the action of U on tensor space, but ignoring the second coordinate.

Clearly the action of U commutes with the action of T_e. It is now enough, for each basis element of $T^r(M_n)$ and for each Hecke algebra generator T_s (corresponding to a simple reflection in the Coxeter group), to establish either that the action of each E_i, F_i, K_i and K_i^{-1} commutes with the action of T_s *or* the action of T_s^{-1} (because $T_s^{-1} = (q^{-1} - 1)T_e + q^{-1}T_s$). A moment's thought shows that it is enough to consider the case $r = 2$, because only simple transpositions are involved.

Denote the typical basis element by $e_{ab} \otimes e_{cd}$ (note that we choose to work with the unquantized basis of $T^r(M_n)$). We first dispose of the easy cases of the K_i and K_i^{-1}. The action of K_i on the basis element is to multiply by v^m, where m is the number of first coordinates in the basis element equal to i. The family of first coordinates is unchanged under the action of $\mathcal{H}$, so the actions commute. The case of K_i^{-1} is similar, but with v^{-m} in place of v^m.

Now we consider the case of E_i. Assume firstly that $b \leq d$. We wish to show that the actions of E_i and T_s commute. However, we know from the proof of Lemma 2.2.1 that the action of T_s on $T^r(M_n)$, when projected to the first coordinates, is similar to the action of T_s on $T^r(V)$, except that the two second coordinates are exchanged and the expression is multiplied by v^{-1} (if $b < d$) or v^0 (if $b = d$), since we are working with unquantized basis elements. We know that the actions of E_i and T_s on tensor space commute. Because the action of U does not change second coordinates of matrix units, we see that the actions of E_i and T_s do indeed commute. Now assume that $b > d$. A similar argument now applies to the actions of E_i and T_s^{-1}. (Here, the second coordinates are exchanged by T_s^{-1} and the expression is multiplied by v.) Hence the actions of E_i and T_s^{-1} commute.

The same techniques as we used for E_i can be used to show that the action of F_i commutes with the action of $\mathcal{H}$.

This completes the proof. ∎

We are now ready to define the candidate for the monomorphism γ of algebras and U-modules from $S_v(n,r)$ into $T^r(M_n)$. Most of the rest of this chapter will be devoted to proving that γ satisfies the required properties.

Recall that $S_v(n,r)$ has a basis denoted by $\{[A]\}$, as A ran over all $n \times n$ matrices with non-negative integer entries summing to r. With each matrix A, we can (uniquely) associate a standard basis element t_A of $T^r(M_n)$, given by

$$e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

exactly as in the classical case. Define $\mathcal{D}_A$ as in the classical case also. Define the integer m_A by

$$m_A := |\{\langle i, j\rangle : 1 \leq i < j \leq r,\ b_i = b_j,\ a_i < a_j\}|.$$

Example Let A be the matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then t_A is given by

$$e_{11} \otimes e_{11} \otimes e_{11} \otimes e_{21} \otimes e_{12} \otimes e_{22} \otimes e_{22},$$

and $m_A = 5$.

Definition 2.2.3

The linear map $\gamma : S_v(n,r) \to T^r(M_n)$ is given by

$$\gamma([A]) := v^{-m_A} t_A . \left(\sum_{w \in \mathcal{D}_A} T_w \right),$$

where the action of $\mathcal{H}(A_{r-1})$ on $T^r(M_n)$ is as before.

Lemma 2.2.4

a) Let

$$t_A := e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

and let

$$t'_A := e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r}$$

be a standard basis element of $T^r(M_n)$ such that $d_i = b_i$ for all i and the ordered pairs $\langle c_i, d_i \rangle$ are a rearrangement of the ordered pairs $\langle a_i, b_i \rangle$. Then the coefficient of t'_A in $\gamma([A])$ is $v^{-m'}$, where

$$m' := |\{\langle i, j \rangle : 1 \leq i < j \leq r,\ d_i = d_j,\ c_i < c_j\}|.$$

b) Any basis element $t = e_{x_1 b_1} \otimes \cdots \otimes e_{x_r b_r}$ occurring with nonzero coefficient in $\gamma([A])$ must satisfy the hypotheses satisfied by t'_A above.

Proof We deal with the proof of a) first.

Firstly, we consider the action of $\sum_{w \in \mathcal{D}_A} T_w$ on $[t_A]$, which of course differs from t_A by some power of v. Consider the action of a typical T_w in the sum, and write

$$w = s_{i_1} \cdots s_{i_m}$$

as a reduced product of simple reflections.

We revert for a moment to acting $\mathcal{S}_r$ on $T^r(M_n)$. Because w is a distinguished coset representative, we find that the matrix units occurring at positions i_j and i_{j+1} in

$$t_j := t_A . s_{i_1} \cdots s_{i_{j-1}},$$

will be in the correct order with respect to the Hebrew lexicographic order (as defined in §1.4.1). (This follows from the fact that $\ell(s_{i_1} \cdots s_{i_{j-1}} s_{i_j}) = \ell(s_{i_1} \cdots s_{i_{j-1}}) + 1$, and the length identities quoted in §1.3.3.) In particular, if

$$t_j = v^* e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

we have $b_{i_j} \leq b_{i_{j+1}}$. (Here, "$*$" denotes some integer whose value is unimportant.) Because of the above remark concerning the Hebrew lexicographic order, we deduce that when computing $t_A . T_w$, we never have to appeal to the case corresponding to $b > d$ in the definition of the action of T_s.

This has repercussions for the quantized case. Recall the remarks in the proof of 2.2.1 about the similarities between the actions of T_s on tensor space and tensor matrix space, in the case $b \leq d$— the action on the second coordinates is essentially permutation. Taking the coefficient of a typical quantized basis element appearing in $[t_A].T_w$, say

$$t := e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r},$$

we now know from the above discussion that if

$$t_A := e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}$$

then the sequence $(d_1, \ldots, d_r)$ is simply the result of acting w on the right of the sequence $(b_1, \ldots, b_r)$ by place permutation.

Let W_J be the parabolic subgroup of $W(A_{r-1})$ which is generated by the $(p, p+1)$ satisfying $b_p = b_{p+1}$. We then see from the above discussion that unless $w \in W_J$, there will certainly be no nonzero coefficient of t'_A occurring in $[t_A].T_w$ (because no basis elements occurring with nonzero coefficient in the latter expression can have their second coordinates in the correct order), and hence in $t_A.T_w$. This simplifies the problem considerably.

The next important observation is that, if $a < b$,

$$e_{ac} \otimes e_{bc}.T_s = v e_{bc} \otimes e_{ac}.$$

Informally, every time an element T_s (where $s \in W_J$) exchanges two matrix units out of Hebrew lexicographic order, the exponent of v is increased by 1. We know that there exists a unique element $u \in W_J \cap \mathcal{D}_A$ such that $t'_A = t_A.u$ (acting as an element of the symmetric group). (Conversely, for any element $u' \in W_J \cap \mathcal{D}_A$, there is a corresponding unique t''_A, with properties similar to those of t'_A.) Write u as a reduced product of simple reflections, $s_{i_1} \cdots s_{i_l}$. Then the entries in the i_k and i_{k+1}-th positions of $t_A.s_{i_1} \cdots s_{i_{k+1}}$ are in Hebrew lexicographic order with different first coordinates and equal second coordinates. Returning to the quantized case (which essentially differs only by powers of v), let the coefficient of t_A in $\gamma([A])$ be C. The number of pairs of matrix units exchanged out of order in passing from t_A to t'_A is $m_A - m'$, where m' is as in the statement of the Lemma. Therefore, we find that the coefficient of t'_A in $C.t_A.\sum_{w \in W_J \cap \mathcal{D}_A} T_w$, and hence in $\gamma([A])$, is $C.v^{m_A - m'}$, by using the above observations. However, we know from the definition of γ (and the correspondence between u' and t''_A mentioned above, in the case where $u' = 1$ and $t''_A = t_A$) that $C = v^{-m_A}$. Thus the coefficient of t'_A is $v^{-m'}$ as claimed.

The proof of b) also follows from the preceding arguments. Let $t = e_{x_1 b_1} \otimes \cdots \otimes e_{x_r b_r}$. We know that the only terms of $\gamma([A])$ which can yield this are of form $t_A.T_w$, where $w \in W_J \cap \mathcal{D}_A$. We have also shown that for such an element w, $t_A.T_w$ is equal to

$$v^* e_{y_1 b_1} \otimes \cdots \otimes e_{y_r b_r},$$

where the $\langle y_i, b_i \rangle$ are some rearrangement of the original $\langle a_i, b_i \rangle$. This completes the proof of b). ∎

Before proceeding with the main theorem, it is worth defining the *relaxed* order, $\preceq$, on the matrix units. We say that $e_{ab} \preceq e_{cd}$ if $(b < d)$ or $(b = d$ and $a > c)$ or $(b = d$ and $a = c)$. A basis element of $T^r(M_n)$ is said to be *relaxed* if the matrix units yielding it appear in it in relaxed order. A basis element is called *H-relaxed* if the matrix units in it appear in Hebrew lexicographic order (for example, t_A for any basis matrix A). These definitions are compatible with the definitions in §1.4.1.

Example The basis element of $T^r(M_n)$ given by

$$e_{21} \otimes e_{11} \otimes e_{11} \otimes e_{11} \otimes e_{22} \otimes e_{22} \otimes e_{12}$$

is relaxed.

One of the main points of Lemma 2.2.4 is that it gives the following result.

Corollary 2.2.5

(i) In the expression $\gamma([A])$, with respect to the standard basis of $T^r(M_n)$, the basis element R_A of $T^r(M_n)$ which is the relaxed rearrangement of t_A is the only relaxed basis element appearing with nonzero coefficient. Furthermore, it appears with coefficient 1.

(ii) In the expression $\gamma([A])$, with respect to the standard basis of $T^r(M_n)$, the basis element t_A is the only H-relaxed basis element appearing with nonzero coefficient. It appears with coefficient v^{-m_A}.

Proof Note that the element R_A is given by

$$\underbrace{e_{n1} \otimes \cdots \otimes e_{n1}}_{A_{n1}} \otimes \cdots \otimes \underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{A_{11}} \otimes \underbrace{e_{n2} \otimes \cdots \otimes e_{n2}}_{A_{n2}} \otimes \cdots \otimes \underbrace{e_{12} \otimes \cdots \otimes e_{12}}_{A_{12}} \otimes \cdots.$$

By definition, any relaxed basis element, R, appearing must have its second coordinates appearing in ascending order. Using Lemma 2.2.4, R must therefore be obtainable from t_A by action on the right by an element $w \in W_J \cap \mathcal{D}_A$, in the notation of the proof of 2.2.4. This means that R satisfies the hypotheses for t'_A in the statement of Lemma 2.2.4. Since R is relaxed, it must be equal to R_A (there is no alternative). We then find that the integer m' defined in the Lemma is equal to 0, thus showing that R_A appears with coefficient 1.

The proof of (ii) uses the same techniques. In this case, $w = 1$ and $m' = m_A$. ∎

Example Let A be the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $R_A = e_{21} \otimes e_{12}$ and

$$\gamma([A]) = e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + (v^2 - 1)e_{22} \otimes e_{11}.$$

Note that the coefficient of R_A in $\gamma([A])$ is 1.

We are now ready to show that γ is an isomorphism of U-modules.

Lemma 2.2.6

(i) The images of the basis $\{[A]\}$ of $S_v(n,r)$ under γ span a U-submodule of $T^r(M_n)$.

(ii) The U-submodule defined above is also an $\mathcal{H}$-submodule, and is the largest $\mathcal{H}$-submodule of $T^r(M_n)$ on which T_w acts as the scalar $q^{\ell(w)}$. (We will refer to this $\mathcal{H}$-submodule as the *ind submodule.*)

(iii) The map γ is an isomorphism of U-modules between $S_v(n,r)$ and the submodule described in (i).

Proof First, we compare $\gamma([A])$ with the element

$$g([A]) := t_A . \left(\sum_{w \in W(A_{r-1})} T_w \right).$$

Using the standard theory of Coxeter groups and Hecke algebras, and denoting by W_A the parabolic subgroup corresponding to $\mathcal{D}_A$, we can rewrite $g([A])$ as

$$t_A . \left(\sum_{u \in W_A} T_u \right) \left(\sum_{w \in \mathcal{D}_A} T_w \right).$$

The point of doing this is that the action of T_u, where $u \in W_A$, on t_A is very simple—it simply multiplies by the scalar $q^{\ell(u)}$. We know (see for example, [H, §3.15]) that the Poincaré polynomial of a Coxeter group of type A_l is

$$\prod_{i=1}^{l} \sum_{k=0}^{i} t^k,$$

and we also know that W_A, being a parabolic subgroup of a Coxeter group of type A, is canonically isomorphic to a direct product of such groups of smaller rank. Its Poincaré polynomial is therefore a product of expressions such as those given above. By substituting $t = q$, we obtain the scalar by which $\sum_{u \in W_A} T_u$ acts, and since $1 + q + \cdots + q^{e-1} \neq 0$ for any natural number e and $\mathbb{Q}(v)$ is a field, we find that this scalar is nonzero. Bringing the quantity v^{-m_A} back into the picture, we find that $\gamma([A])$ and $g([A])$ differ only by multiplication by a nonzero scalar. To prove (i), it is therefore enough to prove that the span of elements $g([A])$ is invariant under the action of U.

Before completing the proof of (i), it is convenient to prove (ii). Lemma 1.6 tells us that

$$\left(\sum_{w \in W} T_w\right) T_d = q^{\ell(d)} \left(\sum_{w \in W} T_w\right).$$

This means that

$$g([A]).T_d = q^{\ell(d)} g([A]),$$

by Lemma 2.2.1, and hence $g([A])$ lies in the ind submodule. We claim that any $t \in T^r(M_n)$ satisfying

$$t.T_w = q^{\ell(w)} t$$

for all $w \in W(A_{r-1})$ lies in the span of the elements $g([A])$. It is easy to see that

$$t = c.t \left(\sum_{w \in W(A_{r-1})} T_w\right)$$

for a certain scalar $c \neq 0$. Let $t_1 = e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}$ be a basis element occurring in t. If we can prove that

$$t_1. \left(\sum_{w \in W(A_{r-1})} T_w\right)$$

lies in the span of the $g([A])$ then the proof of (ii) will follow.

We first deal with the case where $b_1 \leq \cdots \leq b_r$. The proof of Lemma 2.2.4 now shows that $t_1 = v^* t_A.T_w$ for some H-relaxed basis element t_A and some $w \in W(A_{r-1})$. The assertion now follows by using Lemma 1.6. In the case where the sequence of b_i is not increasing, let $(d_1, \ldots, d_r)$ be the rearrangement of $(b_1, \ldots, b_r)$ satisfying $d_1 \leq \cdots \leq d_r$. Choose w such that

$$(d_1, \ldots, d_r).w = (b_1, \cdots, b_r).$$

A simple inductive argument like the one in the proof of Lemma 2.2.4 shows that all elements occurring with nonzero coefficient in $t_1.T_w^{-1}$ are of form

$$e_{x_1 d_1} \otimes \cdots \otimes e_{x_r d_r}.$$

We now use Lemma 1.6 and the previous case (where $b_1 \leq \cdots \leq b_r$) to deduce the required result, thus completing the proof of (ii).

We return to the proof of (i). Let $u \in U$. We know from Lemma 2.2.2 that

$$u.\left(t_A.\sum_{w\in W(A_{r-1})} T_w\right) = (u.t_A).\sum_{w\in W(A_{r-1})} T_w,$$

and this latter element clearly lies in the ind submodule, and hence in the span of the elements $g([A])$. This completes the proof of (i).

For the proof of (iii), we compare the results we get by acting U on $\gamma(S_v(n,r))$ with those in [BLM]. It is enough to verify that the action of u on $\gamma([A])$ agrees with the results of [BLM] where $u \in \{E_*, F_*, K_*, K_*^{-1}\}$ and $[A]$ is a basis element for $S_v(n,r)$. The results we are hoping for are:

a)

$$K_h.\gamma([A]) = v^{s_h}\gamma([A])$$

and

$$K_h^{-1}.\gamma([A]) = v^{-s_h}\gamma([A]),$$

where $s_h := \sum_{k=1}^n a_{hk}$.

b)

$$E_h.\gamma([A]) = \sum_{p\in[1,n]:a_{h+1,p}\geq 1} v^{\beta(p)}[a_{h,p}+1]\ \gamma\left([A+E_{h,p}-E_{h+1,p}]\right),$$

where

$$\beta(p) := \sum_{j>p}(a_{h,j}-a_{h+1,j}).$$

c)

$$F_h.\gamma([A]) = \sum_{p\in[1,n]:a_{h,p}\geq 1} v^{\beta'(p)}[a_{h+1,p}+1]\ \gamma\left([A+E_{h+1,p}-E_{h,p}]\right),$$

where

$$\beta'(p) := \sum_{j<p}(a_{h+1,j}-a_{h,j}).$$

Case a) is easy to verify, because

$$K_h.e_{c_1d_1}\otimes\cdots\otimes e_{c_rd_r} := v^s e_{c_1d_1}\otimes\cdots\otimes e_{c_rd_r},$$

where s is the number of c_i equal to h. By checking coefficients of relaxed terms, via Corollary 2.2.5, we see that the first half of a) holds. Similar techniques work for K_h^{-1}.

To prove case b), choose a basis element $[B]$ which occurs in $E_h.[A]$. It is enough to check that the coefficient of $[B]$ is correct. Because of the way E_h acts on $T^r(M_n)$, we can find a close relationship between the relaxed basis elements R_B and R_A. By using Lemma 2.2.4 b), we see that the only difference is that the rightmost occurrence of $e_{h+1,p}$ in R_A is replaced by $e_{h,p}$ to obtain R_B, for some integer $p : 1 \leq p \leq n$. (If this cannot happen, then the coefficient must be zero as desired.) Let us say that this difference occurs between the P-th component of the basis elements.

We need to identify all basis elements occurring with nonzero coefficients in $\gamma([A])$ whose images under the action of E_h contain a nonzero coefficient of R_B. We do this by considering the action of E_h on $T^r(M_n)$. An example of such an element is R_A itself. A typical basis element R satisfying these hypotheses differs from R_A just to the right of the P-th component; where R_A contains the sequence

$$\cdots \otimes e_{h+1,p} \otimes e_{h,p} \otimes \cdots \otimes e_{h,p} \otimes \cdots,$$

R contains some permutation of this sequence. (Informally, $e_{h+1,p}$ may occur a few places to the right of where it "should".) Denote by R_z the basis element R (as above) where the rightmost occurrence of $e_{h+1,p}$ occurs at place $P+z$ (so in particular, $R_0 = R_A$). Let $h(z)$ denote the multiplicity of $e_{h,p}$ in R_A. (It is equal to $A_{h,p}$.)

We know from Lemma 2.2.4 that R_z occurs in $\gamma([A])$ with coefficient v^{-z}.

It can be shown that

$$\Delta^{r-1}(E_h) = 1 \otimes \cdots \otimes E_h + 1 \otimes \cdots \otimes K_i K_{i+1}^{-1} \otimes E_h + \cdots + E_h \otimes K_i K_{i+1}^{-1} \otimes \cdots \otimes K_i K_{i+1}^{-1},$$

where there are r tensor components in each summand. Using this fact, we find that the contribution of $v^{-z} R_z$ to the coefficient of R_B in the result is v^{x_z}, where

$$x_z := -z + h(z) - z + \left(\sum_{p' > p} A_{h,p'} \right) - \left(\sum_{p' > p} A_{h+1,p'} \right).$$

Summing over all possible z (i.e. $0 \leq z \leq h(z)$), we find that the total contribution from $\gamma([A])$ to R_B is $v^{\beta(p)}[A_{h,p}+1]$, as in the statement of the Lemma.

Case c) is very similar to case b), so we omit the proof.

This completes the proof of the Lemma. ∎

Now that we know that γ is an isomorphism of U-modules, it is not too hard to show that it is an isomorphism of algebras. We know that $S_v(n,r)$ is generated as an algebra by

$$\theta(E_h), \theta(F_h), \theta(K_i), \theta(K_i^{-1}),$$

where $1 \leq h < n$ and $1 \leq i \leq n$. It therefore suffices to show that

$$\gamma(\theta(u) \times [A]) = \gamma(\theta(u)) \times \gamma([A]),$$

for all $[A]$ and for all u in the standard algebra generating set $\{E_*, F_*, K_*, K_*^{-1}\}$ of U.

Theorem 2.2.7 The map γ is a canonical monomorphism of algebras and of U-modules from $S_v(n,r)$ to $T^r(M_n)$.

Proof Recall the remark after Proposition 2.1.3. Again, it is true that the action of $u \in U$ on an element $t \in T^r(M_n)$ is given by

$$u.t = \gamma_r(u) \times t,$$

using the quantized version of the map γ_r appearing in Proposition 2.1.3. We claim that it is enough to show that $\gamma_r(u) = \gamma(\theta(u))$ for elements u in the standard generating set. For if this is true, Lemma 2.2.6, the preceding remark, and the fact that

$$u.[A] := \theta(u) \times [A]$$

show that

$$\gamma(\theta(u) \times [A]) = \gamma(u.[A]) = u.\gamma([A]) = \gamma_r(u) \times \gamma([A]) = \gamma(\theta(u)) \times \gamma([A]),$$

as required.

The case $u = K_i$ is fairly easy, so we tackle this first. Beilinson et al. [BLM] define

$$\theta(K_i) = \sum_{D \in \mathbb{D}_r} v^{d_i}[D],$$

where $\mathbb{D}_r$ is the set of $n \times n$ diagonal matrices with nonnegative integer entries summing to r. Calculating $\gamma([D])$, where D is a diagonal matrix, turns out to be remarkably simple, because $m_D = 0$ and because the Hecke algebra $\mathcal{H}(A_{r-1})$ acts via permutations! The reason for the latter is as follows. Consider a typical element $T_w \in \mathcal{D}_D$, and write $w = s_{i_1} \cdots s_{i_l}$ as a reduced expression, as usual. Let

$$t_j := t_D.s_{i_1} \cdots s_{i_{j-1}}.$$

As in the proof of Lemma 2.2.4, the matrix units occurring at positions i_j and i_{j+1} in t_j are lexicographically ordered. However, if $a < b$,

$$e_{aa} \otimes e_{bb}.T_s = e_{bb} \otimes e_{aa},$$

so we see that $\mathcal{H}$ acts only via permutations. From this, we see that

$$\gamma(\theta(K_i)) = \sum_D \left(v^{d_i}.t_D.\left(\sum_{w\in\mathcal{D}_D} T_w \right)\right),$$

where the outer sum ranges over all possible values of the diagonal matrices $D = \mathrm{diag}(d_1, \ldots, d_r)$ subject to the constraint $\sum d_i = r$.

We now check that this is equal to $\gamma_r(K_i)$. It is not hard to show that

$$\Delta^{r-1}(K_i) = \underbrace{K_i \otimes \cdots \otimes K_i}_{r \text{ times}}.$$

We know that

$$\theta_1(K_i) = \sum_{j=1}^{n} v^{\delta(i,j)} e_{j,j}.$$

Using the fact that

$$\gamma_r := \underbrace{(\theta_1 \otimes \cdots \otimes \theta_1)}_{r \text{ times}} \Delta^{r-1},$$

it is fairly easy to see that the maps γ_r and $\gamma\theta$ agree on K_i, as required.

The case $u = K_i^{-1}$ follows by an argument entirely analogous to the previous one.

We now tackle the case of $u = E_h$. Following [BLM, §5.7], we have

$$\theta(E_h) := \sum_{D\in\mathbb{D}_{r-1}} [E_{h,h+1} + D].$$

We then use a similar technique to the one used in the K_i case to argue that $\mathcal{H}$ "almost" acts via permutations (more precisely—it does act via permutations, but introduces certain powers of v). The situation is very similar to the K_i situation, *except* that

$$e_{hh+1} \otimes e_{h+1h+1}.T_s = v e_{h+1h+1} \otimes e_{hh+1}$$

and

$$e_{hh} \otimes e_{hh+1}.T_s = v e_{hh+1} \otimes e_{hh}.$$

Informally, this means that the situation is similar to the classical case, except the processes of passing an e_{h+1h+1} to the left of the e_{hh+1} and that of passing an e_{hh} to the right of an e_{hh+1} multiply the relevant basis element by v. However, we know from Corollary 2.2.5 that "relaxed" basis elements have coefficient 1. Such basis elements contain $e_{h,h+1}$ to the right of any possible occurrences of e_{hh} and e_{h+1h+1}. This means that the coefficient of any typical basis element of $T^r(M_n)$ occurring in $\gamma(\theta(E_h))$, say

$$t = e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r},$$

occurs with coefficient v^s, where s is the number of e_{hh} occurring to the right of the occurrence of e_{hh+1} minus the number of e_{h+1h+1} occurring to the right of the occurrence of e_{hh+1}.

We know that

$$\Delta^{r-1}(E_h) = 1 \otimes \cdots \otimes E_h + 1 \otimes \cdots \otimes K_i K_{i+1}^{-1} \otimes E_h + \cdots + E_h \otimes K_i K_{i+1}^{-1} \otimes \cdots \otimes K_i K_{i+1}^{-1},$$

where there are r tensor components in each summand. Since $\theta_1(E_h) = e_{hh+1}$ and we know what the $\theta_1(K_i^{\pm 1})$ are, it is possible to see from the above discussion that the maps γ_r and $\gamma\theta$ agree on E_h.

The case of $u = F_h$ is entirely analogous to that of $u = E_h$.

This completes the proof of the main result. ∎

Example Consider $S_v(2,2)$. Let

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let I be the 2×2 identity matrix. Let e_A and e_I be the corresponding elements in the Dipper-James basis of $S_v(2,2)$. We know from properties of the Hecke algebra of type A_1 embedded in $S_v(2,2)$ that $e_A^2 = qe_I + (q-1)e_A$, but we wish to show this directly. It can be checked that $v[A] = e_A$ and $[I] = e_I$.

Calculating the values of γ on these basis elements, we find that

$$v.\gamma([A]) = ve_{12} \otimes e_{21} + ve_{21} \otimes e_{12} + (v^2 - 1)e_{22} \otimes e_{11},$$

and that

$$\gamma([I]) = e_{11} \otimes e_{22} + e_{22} \otimes e_{11}.$$

Squaring $v.\gamma([A])$ in the ring $T^r(M_n)$, we find that it gives

$$v^2 e_{11} \otimes e_{22} + v^2 e_{22} \otimes e_{11} + (v^2-1)^2 e_{22} \otimes e_{11} + (v^3 - v)e_{21} \otimes e_{12} + (v^3 - v)e_{12} \otimes e_{21}.$$

The relaxed basis elements occurring in these expressions are $e_{11} \otimes e_{22}$, corresponding to $[I]$, and occurring with coefficient $q = v^2$, and $e_{21} \otimes e_{12}$, corresponding to $[A]$ and occurring with coefficient $(v^3 - v) = (q-1)v$. Hence we have

$$(v[A])^2 = q[I] + (q-1)(v[A]),$$

thus proving the desired result. ∎

Remarks on Theorem 2.2.7 As can be seen from the example, these methods give a fairly easy way of computing products in $S_v(n,r)$, without having to resort to difficult procedures involving flags and finite fields.

Since the classical case corresponding to this (i.e. when q and v are replaced by 1) gives the situation where the classical Schur algebra is the r-th symmetric power of the $n \times n$ matrix algebra, the Theorem casts more light on the nature of the quantization. The space $T^r(M_n)$ provides an arena in which the quantization takes place.

It should be noted that this product rule also gives a product rule for $S_q(n,r)$ in terms of the more usual basis $\{e_A\}$. In [BLM, Lemma 2.3], it is proved that $e_A = v^{s_A}[A]$, where

$$s_A := \sum_{1 \leq i,j,k,l \leq n} A_{i,j} A_{k,l},$$

where the sum is taken with the restrictions that $i \geq k$ and $j < l$. This exhibits a close connection between the two bases. The restrictions on the sum are reminiscent of the relaxed order, $\preceq$, defined on the matrix units.

The product rule we obtain from using the above Theorem is of a convenient form for proving theorems whose classical analogues are proved using Schur's Product Rule.

2.3. On certain subalgebras of the q-Schur algebra

Using the description of the v-Schur algebra given in §2.2, we can describe a (fairly large) family of subalgebras of the q-Schur algebra which are spanned by certain natural basis elements e_A. To do this, we introduce the notion of a "good" set of matrix units.

Definition A subset G of $\{e_{ij} : 1 \leq i,j \leq n\}$ is called a *good* set of matrix units if it satisfies the following two conditions:

(i) G spans a subalgebra of the full matrix ring M_n;

(ii) if $e_{ab} \in G$ and $e_{cd} \in G$ where $a < c$ and $b > d$, then also $e_{ad} \in G$ and $e_{cb} \in G$.

Lemma 2.3.1 Let G be a good set of matrix units. Then $T^r(G)$ is a subalgebra of $T^r(M_n)$, and a right $\mathcal{H}$-submodule of $T^r(M_n)$.

Proof It is clear that $T^r(G)$ is a subalgebra of $T^r(M_n)$ because the elements of G span a subalgebra of M_n.

We now need to check that if $[e_{i_1 j_1} \otimes \cdots \otimes e_{i_r j_r}] \in T^r(G)$, then acting T_s (where $s = (p, p+1)$ for some p) on this element produces another element of $T^r(G)$. Because of the way the multiplication is defined, we need only check this in the case $r = 2$.

We now consider the matrix units appearing in the expression for $[e_{ab} \otimes e_{cd}].T_s$ (where $s = (12)$) as in the definition of the action of $\mathcal{H}$ on $T^r(M_n)$. It is easily checked that if $a \le c$ and $b \le d$, then no new matrix units can appear in the result. It is similarly easily checked that if $a > c$ and $b > d$, no new matrix units can appear in the result. If $a > c$ and $b = d$, or if $a = c$ and $b > d$, similar remarks hold.

We are left with the two most interesting cases, namely the case when $a > c$ and $b < d$, and the case when $a < c$ and $b > d$. In both cases, we find that e_{ad} and e_{cb} are also elements of G, from the definition of a good set of matrix units. It is now clear from the definition of the action of $\mathcal{H}$ that all matrix units appearing in the result are elements of G, as desired. ∎

Definition If G is a good set of matrix units, we denote by $S_v(G)$ the subspace of $S_v(n, r)$ spanned by all basis elements $[A]$ where A is a matrix such that $A_{ij} \neq 0 \Rightarrow e_{ij} \in G$. We denote by $S_q(G)$ the corresponding subspace of $S_q(n, r)$ spanned by elements e_A.

Lemma 2.3.2 Using the monomorphism γ defined in §2.2, the following identity holds in $T^r(M_n)$:

$$\gamma(S_v(n, r)) \cap T^r(G) = \gamma(S_v(G)).$$

Proof We first establish that $\gamma(S_v(G)) \le \gamma(S_v(n, r)) \cap T^r(G)$. Clearly $S_v(G) \le S_v(n, r)$. It is now enough to show that $\gamma([A]) \in T^r(G)$ where $[A]$ is a basis element of $S_v(G)$. By definition of γ, $\gamma([A])$ is of the form $t_A.h$ for some $h \in \mathcal{H}$. By the construction of $[A]$, we find that $t_A \in T^r(G)$, and therefore $\gamma([A]) \in T^r(G)$ by Lemma 2.3.1.

Now consider a typical element in $\gamma(S_v(n, r)) \cap T^r(G)$ given by

$$g = \sum_A c_A \gamma([A]),$$

where the c_A are elements of $\mathbb{Q}(v)$. We claim that unless $[A]$ is a basis element of $S_v(G)$, c_A must be zero. This is achieved by considering relaxed basis elements occurring in the expression for g. These relaxed basis elements must all lie in $T^r(G)$. It follows from the definitions that the relaxed basis element corresponding to $[A]$ is an element of $T^r(G)$ if and only if $[A]$ is a basis element for $S_v(G)$. This establishes the reverse inclusion. ∎

Theorem 2.3.3 $S_v(G)$ is a subalgebra of $S_v(n,r)$ and $S_q(G)$ is a subalgebra of $S_q(n,r)$. The dimension of each subalgebra is

$$\binom{m+r-1}{r},$$

where m is the cardinality of the set G.

Proof We know that $S_v(G)$ is a subspace of $S_v(n,r)$ and it is clearly a subalgebra by Lemma 2.3.2, where it was shown to be an intersection of two subalgebras of $T^r(M_n)$. The corresponding result for $S_q(G)$ follows immediately by consideration of products $e_A e_{A'}$ where both elements are in $S_q(G)$.

By construction of the basis for $S_v(G)$ or $S_q(G)$, we see that the dimension is equal to the number of compositions of r into m pieces. The result now follows from elementary combinatorics.

∎

Examples

We now present some examples of good sets of matrix units.

1. Let

$$G = \{e_{ij} : 1 \leq i \leq j \leq n\}.$$

This spans the subalgebra of M_n given by upper triangular matrices. It is easily checked that G is good, and that $S_q(G)$ is the Borel subalgebra S^+.

2. Choose a finite sequence of integers r_i satisfying $0 = r_0 < r_1 < \ldots < r_s = n$. Then let

$$G = \{e_{ij} : 1 \leq i, j \leq n \text{ and } (r_h < i \leq r_{h+1} \iff r_h < j \leq r_{h+1})\}.$$

This spans the subalgebra of M_n given by matrices of a particular block form (depending on the sequence r_i). The set G is good, and the algebra $S_q(G) \leq S_q(n,r)$ exists.

3. Let

$$G = \{e_{11}\} \cup \{e_{ij} : 2 \leq i \leq n, 1 \leq j \leq n\}.$$

This spans a subalgebra of M_n. Furthermore, G is good and the algebra $S_q(G) \leq S_q(n,r)$ exists.

3. A Straightening Formula for Quantized Codeterminants

In this chapter, we introduce q-analogues of codeterminants, which were defined in the classical case by J.A. Green [G2, §4]. These were shown to be parametrised by pairs of row-semistandard tableaux of shape λ, where $\lambda \in \Lambda(n, r)$. If both the tableaux are standard, and the composition λ is dominant, then the codeterminant is said to be standard. The set of standard codeterminants forms a free $\mathbb{Z}$-basis for the integral form of the classical Schur algebra. An algorithm for expressing an arbitrary codeterminant as a sum of standard ones has been described in [W].

The main results of this chapter are Theorem 3.2.6 and Corollary 3.2.7, which prove constructively that the set of standard q-codeterminants forms a free $\mathbb{Z}[q, q^{-1}]$-basis of the q-Schur algebra, $S_q(n, r)$. This fact has several important applications later in the thesis, and the main results in §4, §5 and §6 rely on it fairly heavily.

3.0. Conventions and Review of §2

We will write v^* to mean some integer power of v. Usually no attempt will be made to keep track of the value.

In some of the proofs in this chapter, we will use the technique of proving that an element $w \in W$, where W is a Coxeter group, satisfies a certain property P by "induction on left subwords of w." The idea is as follows. Let $s_{i_1} \cdots s_{i_m}$ be any fixed reduced expression for w. We then prove by induction on p (where $1 \leq p \leq m$) that the element w_p defined by $s_{i_1} \cdots s_{i_p}$ satisfies the property P; this then proves that w has the required property. It is immediate from the properties of Coxeter groups that if w is a distinguished right coset representative with respect to some parabolic subgroup of W, then so are all its left subwords. It is also immediate that if w lies in some parabolic subgroup W_P of W, then so do all its left subwords.

It is convenient to recall the action of $\mathcal{H}$ on the right of $T^r(M_n)$ given in §2.2, but this time with respect to a slightly different basis.

We now recall the *quantized basis* of $T^r(M_n)$. This was introduced in §2.2. We now re-express this as the set $\{[u_{i,j}] : i, j \in I(n, r)\}$. Here, $[u_{i,j}]$ is defined to be $v^{-m(i)-m(j)} u_{i,j}$, where the function m is as in §1.3.6.

It was shown in Lemma 2.2.1 that $T^r(M_n)$ admits a right $\mathcal{H}$-module structure which we now recall. Let $T_s = T_{(p,p+1)}$. Then:

$$[u_{i,j}].T_s = \begin{cases} q[u_{i.s,j.s}] & \text{if } i_p \leq i_{p+1} \text{ and } j_p \leq j_{p+1}; & (1) \\ [u_{i.s,j.s}] + (q-1)[u_{i,j}] & \text{if } i_p > i_{p+1} \text{ and } j_p > j_{p+1}; & (2) \\ [u_{i.s,j.s}] + (q-1)[u_{i,j.s}] & \text{if } i_p > i_{p+1} \text{ and } j_p \leq j_{p+1}; & (3) \\ q[u_{i.s,j.s}] + (q-1)[u_{i,j}] - (q-1)[u_{i,j.s}] & \text{if } i_p \leq i_{p+1} \text{ and } j_p > j_{p+1}. & (4) \end{cases}$$

From this, the action of $T^{-1}_{(p,p+1)}$ immediately follows. It is given by:

$$[u_{i,j}].T_s^{-1} = \begin{cases} q^{-1}[u_{i.s,j.s}] & \text{if } i_p \geq i_{p+1} \text{ and } j_p \geq j_{p+1}; & (5) \\ [u_{i.s,j.s}] + (q^{-1}-1)[u_{i,j}] & \text{if } i_p < i_{p+1} \text{ and } j_p < j_{p+1}; & (6) \\ [u_{i.s,j.s}] + (q^{-1}-1)[u_{i,j.s}] & \text{if } i_p < i_{p+1} \text{ and } j_p \geq j_{p+1}; & (7) \\ q^{-1}[u_{i.s,j.s}] + (q^{-1}-1)[u_{i,j}] - (q^{-1}-1)[u_{i,j.s}] & \text{if } i_p \geq i_{p+1} \text{ and } j_p < j_{p+1}. & (8) \end{cases}$$

Note 3.0.1

Notice that this action specialises (when q and v are replaced by 1) to permutation, and that any basis element occurring with nonzero coefficient in $u_{a,b}.T_w$ is of form $u_{c,d}$ where $a \sim c$ and $b \sim d$.

The rest of this chapter falls into two main sections. In §3.1, we show that any q-codeterminant can be expressed as a $\mathbb{Z}[q, q^{-1}]$-linear sum of codeterminants of dominant shape. In §3.2, we prove a quantum analogue of the straightening formula for codeterminants. This is a procedure for expressing any codeterminant of dominant shape as a $\mathbb{Z}[q, q^{-1}]$-linear sum of standard codeterminants. This was proved by Woodcock [W] in the classical case. We use some of the techniques in that paper to prove the quantum analogue, but some steps are considerably more delicate and require new methods.

3.1. Hecke algebras and codeterminants of dominant shape

Lemma 3.1.1 Let $t = u_{a,b}$ (where $a, b \in I(n,r)$) be a basis element of $T^r(M_n)$ of parabolic form. Then

$$y := t.\frac{x_W}{P_t}$$

lies in $S_v(n,r)$. Moreover, if t is expressed as an $\mathcal{A}$-linear sum of elements $\xi_{i,j}$, then all elements occurring with nonzero coefficients in the sum satisfy $i \sim a$ and $j \sim b$.

Proof Lemma 1.6 shows that

$$T_d x_W = q^{\ell(d)} x_W,$$

where W is the Coxeter group of type A_{r-1}. We now find that y lies in the ind submodule of $T^r(M_n)$, and hence in $S_v(n,r)$ by Lemma 2.2.6. This proves that t is a $\mathbb{Q}(v)$-linear combination of elements $\xi_{i,j}$. Thanks to Corollary 2.2.5, it is now enough to show that the coefficients of the relaxed basis elements occurring in y lie in $\mathcal{A}$, because that will force the coefficient of each $\xi_{i,j}$ to lie in $\mathcal{A}$.

Define W_J to be the parabolic subgroup of W generated by the $(p, p+1)$ satisfying $b_p = b_{p+1}$. Let t_0 be the unique element of form $u_{a',b}$ satisfying $(a', b) \sim (a, b)$ and the condition that if $b_p = b_{p+1}$ then $a'_p \leq a'_{p+1}$. Let W_I be the parabolic subgroup of $\mathcal{S}_r$ stabilising t_0. There exists a unique $w \in \mathcal{D}_I \cap W_J$ such that $a'.w = a$. Then we see by induction on left subwords of w that $u_{a',b}.T_w = v^* u_{a,b}$, as required. By substituting t_0 for t in the definition of y, and by using the observation that if $a \in \mathcal{A}$ then $v^m a \in \mathcal{A}$ for any $m \in \mathbb{Z}$, we find that it is enough to prove the lemma in the case where $t = t_0$, because $P_t = P_{t_0}$.

We now assume $t = t_0$ and define the parabolic subgroup W_I to be generated by all $(p, p+1)$ such that $a'_p = a'_{p+1}$ and $b_p = b_{p+1}$. Let $\mathcal{D}_I$ be the set of distinguished right coset representatives corresponding to W_I. Then, using the theory of Coxeter groups, we have

$$x_W = x_I \left(\sum_{u \in \mathcal{D}_I} T_u \right).$$

We know (see for example, [H, §3.15]) that the Poincaré polynomial of a Coxeter group of type A_l is

$$\prod_{i=1}^{l} \sum_{k=0}^{i} z^k,$$

and we also know that W_I, being a parabolic subgroup of a Coxeter group of type A, is canonically isomorphic to a direct product of such groups of smaller rank. Its Poincaré polynomial is therefore a product of expressions such as those given above. By substituting $z = q$, we obtain the scalar by which $\sum_{u \in W_I} T_u$ acts, and we find that

$$t.x_I = P_t.t.$$

This means that

$$y = t. \left(\sum_{u \in \mathcal{D}_I} T_u \right),$$

and hence, since the action of $\mathcal{H}$ on $T^r(M_n)$ is defined over $\mathcal{A}$, that the coefficients of any basis elements occurring in y, in particular the relaxed ones, lie in $\mathcal{A}$.

The last assertion follows from Corollary 2.2.5 and Note 3.0.1. ∎

Definition Let $i, j \in I(n, r)$ be given by $i = (i_1, \ldots, i_r)$ and $j = (j_1, \ldots, j_r)$. Then the element $\tau_{i,j} \in T^r(M_n)$ is defined as

$$\tau_{i,j} := u_{i,j}.\frac{x_W}{P_{i,j}}.$$

If j is of parabolic form, we know from Lemma 3.1.1 that $\tau_{i,j}$ can be considered to lie in $S_v(n, r)$.

Note that if the $u_{i,j}$ constructed above is H-relaxed, then $\tau_{i,j}$ is equal to $\xi_{i,j}$ multiplied by a certain power of v.

Also note that Lemma 3.1.1 tells us that each element $\tau_{i,j}$ is of form

$$\sum c_{i',j'}\xi_{i',j'},$$

where the elements $c_{i',j'}$ lie in $\mathcal{A}$, and the indices i' and j' satisfy $i' \sim i$ and $j' \sim j$.

During the proofs that follow, we shall need a way of testing whether two elements of $\mathcal{H}$ are equal to each other. To do this, we construct a right $\mathcal{H}$-module M with the property that there exist certain elements $m \in M$ such that if $h_1, h_2 \in \mathcal{H}$ and $m.h_1 = m.h_2$ then $h_1 = h_2$. By linearity, it is enough to show that if $m.h = 0$ then $h = 0$.

Lemma 3.1.2 Let V be a $\mathbb{Q}(v)$-vector space of dimension n with basis $e_1, \ldots, e_n$. We make $M := V^{\otimes r}$ into a right $\mathcal{H}$-module in the usual way (i.e. via the tensor space action defined in §1.3.6). Let $h \in \mathcal{H}$. Then:

(i) if $m := e_1 \otimes \cdots \otimes e_r$, then $m.h = 0 \Rightarrow h = 0$;

(ii) if $m_\pi := e_{\pi(1)} \otimes \cdots \otimes e_{\pi(r)}$, where $\pi \in \mathcal{S}_r$, then $m_\pi.h = 0 \Rightarrow h = 0$.

Proof We first prove (i). Let m and m_π be defined as in the statement of the Lemma. By using induction on $\ell(\pi)$, we find that $m.T_\pi = v^{\ell(\pi)}m_\pi$. If $\pi \neq \pi'$ then $m_\pi \neq m_{\pi'}$; in fact, the set $\{m_\pi : \pi \in \mathcal{S}_r\}$ is linearly independent in $V^{\otimes r}$. However, $\{T_w : w \in \mathcal{S}_r\}$ is known to be a basis for $\mathcal{H}$. Since v is invertible, we now find that if $m.h = 0$, then $h = 0$.

We now prove (ii). We know from the above argument that $m.T_\pi = v^{\ell(\pi)}m_\pi$. Assuming that $m_\pi.h = 0$, then $m.(v^{-\ell(\pi)}T_\pi h) = 0$. Now (i) shows that $v^{-\ell(\pi)}T_\pi h = 0$, and, since v and T_π are both invertible, that $h = 0$. This completes the proof. ∎

We now need to introduce a family of bases for $\mathcal{H}$, of which the usual basis $\{T_w\}$ is an example. The motivation behind doing this is that by picking the correct basis, it is possible to deduce certain results concerning the coefficients of basis elements in the $\tau_{i,j}$ which were introduced earlier. This will be important later.

For convenience, we now choose, once and for all, a "favourite" reduced expression $\rho(w)$ for each element $w \in W$.

Definition Let $j \in I(n,r)$ be of parabolic form, let W_J be its associated parabolic subgroup, and let $\mathcal{D}_J$ be the associated set of distinguished right coset representatives. To each element $w \in \mathcal{D}_J$, we will associate an element $D^j_w \in \mathcal{H}$ which specialises as q is replaced by 1 to w in the group algebra.

Suppose $\rho(w) = s_{i_1} \cdots s_{i_m}$. Earlier, we showed how $\mathcal{S}_r$ acts on $I(n,r)$ on the right. For each integer s such that $1 \leq s \leq m$ we define an element h_s of $\mathcal{H}$ as follows. Consider $j' := j.s_{i_1} \cdots s_{i_{s-1}}$. Because w is a distinguished right coset representative, it must be the case that $j'_{i_s} \neq j'_{i_{s+1}}$. (This follows from the length identities in §1.3.3, the definition of distinguished right coset representatives, and the fact that any left segment of a distinguished right coset representative is another distinguished right coset representative.) If $j'_{i_s} > j'_{i_{s+1}}$ then $h_s := T_{i_s}^{-1}$, otherwise $h_s := T_{i_s}$. We then define D_w^j to be $h_1 h_2 \cdots h_m$.

Remark

If $1 \leq a < b \leq n$, then $e_b \otimes e_a.T_s^{-1} = v^{-1} e_a \otimes e_b$ and $e_a \otimes e_b.T_s = v e_b \otimes e_a$. Define, for any $i \in I(n,r)$,

$$m(i) := |\{\langle a, b\rangle : 1 \leq a < b \leq r,\ i_a > i_b\}|.$$

(For example, if $i = (\pi(1), \ldots, \pi(r))$ as in the statement of Lemma 3.1.2 (ii), then $m(i) = \ell(\pi)$.)

If $s = (p, p+1)$ and $j_p \leq j_{p+1}$ then $u_{i,j}.T_s$ is a linear combination of elements $u_{i',j.s}$ (see §3.0, (1) and (3)). If $j_p \geq j_{p+1}$ then $u_{i,j}.T_s^{-1}$ is a linear combination of elements $u_{i',j.s}$ (see §3.0, (5) and (7)).

The crucial property of the element D_w^j is as follows:

$$v^{m(j)} e_{j_1} \otimes \cdots \otimes e_{j_r}.D_w^j = v^{m(j.w)} e_{w(j_1)} \otimes \cdots \otimes e_{w(j_r)}.$$

This can be seen by induction on left subwords of w.

It is also follows that basis elements occurring with nonzero coefficients in $u_{i,j}.D_w^j$ are of form $u_{i',j.w}$.

Example

Let $j = (4,4,1,2)$. Then the parabolic subgroup W_J is generated by $s_1 = (1,2)$. A routine check shows that the element $w \in \mathcal{S}_4$ given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

lies in $\mathcal{D}_J$. The element $j.w \in I$ is equal to $(2,4,1,4)$. A reduced expression $\rho(w)$ for w is given by $s_3 s_2 s_1 s_3$. The element $D_w^j \in \mathcal{H}$ is now given by $T_3 T_2^{-1} T_1^{-1} T_3^{-1}$. (It will be seen later that the element D_w^j is independent of the reduced expression chosen for w.)

We now compare the elements D^j_w with their counterparts T_w. To do this, it is convenient to define an element $R(j) \in I(r,r)$ (N.B.—not necessarily in $I(n,r)$) which is in some sense a "refinement" of j. We will then act various D^j_w on this $R(j)$, and appeal to Lemma 3.1.2 to relate them to the usual basis consisting of the T_w.

Definition Let $i \in I(n,r)$ be of parabolic form. Then the element $R(i)$ is the unique element of $I(r,r)$ satisfying the following properties:

(i) $R(i) = (\pi(1), \ldots, \pi(r))$ for some $\pi \in \mathcal{S}_r$;

(ii) if $1 \leq a < b \leq r$ and $i_a < i_b$ (respectively $i_a > i_b$) then $R(i)_a < R(i)_b$ (respectively $R(i)_a > R(i)_b$);

(iii) if $1 \leq a < b \leq r$ and $i_a = i_b$ then $R(i)_a < R(i)_b$.

Example Let $n = 4, r = 8$ and $i = (1,1,1,4,4,4,3,3)$. Then $R(i) = (1,2,3,6,7,8,4,5)$.

Remarks Note that $m(R(i)) = m(i)$ for any $i \in I(n,r)$ of parabolic form. Also note that

$$v^{m(R(i))} e_{R(i)_1} \otimes \cdots \otimes e_{R(i)_r}.D^i_w = v^{m(R(i).w)} e_{R(i)_{w(1)}} \otimes \cdots \otimes e_{R(i)_{w(r)}}.$$

Together with Lemma 3.1.2, this shows that the element D^j_w is in fact independent of the reduced expression we were using for w.

We can say more than this: if W_I is the parabolic subgroup associated with i, and $u \in W_I$, then

$$v^{m(R(i))} e_{R(i)_1} \otimes \cdots \otimes e_{R(i)_r}.T_u D^i_w = v^{m(R(i).uw)} e_{R(i)_{uw(1)}} \otimes \cdots \otimes e_{R(i)_{uw(r)}}.$$

This follows from the definitions of $R(i)$ and D^i_w: all the actions are essentially permutations, but with powers of v introduced to keep track of the changing value of the m function (which can be thought of as "length").

Lemma 3.1.3 Let $j \in I(n,r)$ be of parabolic form, with associated parabolic subgroup W_J. Then:

(i)
$$\left(\sum_{u \in W_J} T_u\right)\left(\sum_{w \in \mathcal{D}_J} D^j_w\right) = q^{-m(j)} \sum_{w \in W} T_w.$$

(ii) The set
$$\{T_u D^j_w : u \in W_J, w \in \mathcal{D}_J\}$$
is a basis for $\mathcal{H}$.

Proof Let $\pi \in \mathcal{S}_r$ be such that $R(j) = (\pi(1), \dots, \pi(r))$. Then

$$e_1 \otimes \cdots \otimes e_r . T_\pi = v^{m(R(j))} e_{R(j)_1} \otimes \cdots \otimes e_{R(j)_r}.$$

It now follows that

$$v^{m(R(j))} e_{R(j)_1} \otimes \cdots \otimes e_{R(j)_r} . T_\pi^{-1} . \left(\sum_{w \in W} T_w \right) = \sum_{w \in W} v^{\ell(w)} e_{w(1)} \otimes \cdots \otimes e_{w(r)}.$$

It also follows from the preceding remarks that

$$v^{m(R(j))} e_{R(j)_1} \otimes \cdots \otimes e_{R(j)_r} . \left(\sum_{u \in W_J} T_u \right) \left(\sum_{w \in \mathcal{D}_J} D_w^j \right) = \sum_{w \in W} v^{\ell(w)} e_{w(1)} \otimes \cdots \otimes e_{w(r)}.$$

Applying Lemma 3.1.2 (ii), this shows that

$$\left(\sum_{u \in W_J} T_u \right) \left(\sum_{w \in \mathcal{D}_J} D_w^j \right) = T_\pi^{-1} . \left(\sum_{w \in W} T_w \right),$$

where the right-hand side is known to be equal to

$$q^{-\ell(\pi)} \sum_{w \in W} T_w,$$

thus establishing (i) since $\ell(\pi) = m(R(j)) = m(j)$.

For the proof of (ii), note that the set

$$\{v^{m(R(j))} e_{R(j)_1} \otimes \cdots \otimes e_{R(j)_r} . T_u D_w^j : u \in W_J, w \in \mathcal{D}_J\}$$

is linearly independent in $T^r(V)$, because each element is a power of v times a certain basis element, and basis elements corresponding to distinct elements are nontrivial permutations of each other. Lemma 3.1.2 (ii) now shows that the set

$$\{T_u D_w^j : u \in W_J, w \in \mathcal{D}_J\}$$

is independent in $\mathcal{H}$, and, since it has the correct cardinality, that it is a basis for $\mathcal{H}$. ∎

Definition Let $z \in I(n,r)$. Define z^+ to be the element of $I(n,r)$ satisfying $z_i^+ = f(z_i)$ for all i and for $f \in \mathcal{S}_n$ of minimal length subject to the condition that $\mathrm{wt}(z^+) \in \Lambda^+(n,r)$. (Such an f is uniquely defined, as will be illustrated in the following example.)

Example Let $n = 5$, $r = 9$ and $z = (1,1,2,2,2,2,4,4,4)$, so that $\mathrm{wt}(z) = (2,4,0,3,0)$. Then the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

works in the above definition of f, and $z^+ = (3,3,1,1,1,1,2,2,2)$, with $\mathrm{wt}(z^+) = (4,3,2,0,0)$. Note that $\mathrm{wt}(z)$ is a permutation of $\mathrm{wt}(z^+)$, and that $\mathrm{wt}(z) \in \Lambda^+$. This will always be the case, and in fact $\mathrm{wt}(z^+) = f.\mathrm{wt}(z)$. There is exactly one such f of minimal length, showing why f and z^+ are well-defined.

Lemma 3.1.4 Let $z \in I(n,r)$ be such that $i < j \Rightarrow z_i \leq z_j$. In particular, z is of parabolic form; let the associated parabolic subgroup be W_Z. Then:

(i) $\tau_{z,z^+} \times \tau_{z^+,z} = v^* \tau_{z,z}$;

(ii) There exists $u \in S_v(n,r)$ satisfying $u.\xi_{z^+,z} = v^* \xi_z$.

Proof It is immediate that if z is of parabolic form, then so is z^+, and the stabiliser of z^+ is precisely W_Z. We then know from the remarks following the definition of the $\tau_{i,j}$ that $\tau_{z^+,z} = v^* \xi_{z^+,z}$, for some integer $*$, and that $\tau_{z,z} = v^* \xi_{z,z}$ for some integer $*$. Furthermore, we know that τ_{z,z^+} corresponds to some element in $S_v(n,r)$. This establishes the proof of (ii) assuming (i), so it remains to prove (i).

To prove (i), it is enough, by the results of Corollary 2.2.5, to prove that the only relaxed basis element occurring in $\tau_{z,z^+} \times \tau_{z^+,z}$ is that corresponding to $\tau_{z,z}$, and that it occurs with a coefficient which is a power of v.

The only relaxed basis element occurring with nonzero coefficient in $\tau_{z^+,z}$ which can contribute to a relaxed basis element in the product is $u_{z^+,z}$. This follows from Corollary 2.2.5 and the fact that $\tau_{z^+,z} = v^*.\xi_{z^+,z}$. Lemma 2.2.4 tells us that the coefficient of $u_{z^+,z}$ is a power of v.

The problem now reduces further—it is now enough to show that if u_{a,z^+} occurs in τ_{z,z^+} with nonzero coefficient then $a = z$ and the coefficient is a power of v.

Using Lemma 3.1.3 (i), the definition of τ, and the action of T_y (where $y \in W_Z$) on u_{z,z^+}, we find that

$$\tau_{z,z^+} = v^*.u_{z,z^+}.\left(\sum_{w \in \mathcal{D}_Z} D_w^{z^+}\right).$$

From our study of the properties of $D_w^{z^+}$, we see that any basis element occurring with nonzero coefficient in $u_{z,z^+} D_w^{z^+}$, where w is as in the above sum, is of form $u_{b,z^+.w}$. The implication of this is that the only basis elements of the desired form come from the case $w = 1$ in the sum. This is enough to complete the proof of the Lemma. ∎

Remark on Lemma 3.1.4 In the classical case, this result is obvious (see [W]).

Proposition 3.1.5 Any codeterminant is a $\mathbb{Z}[q,q^{-1}]$-linear combination of codeterminants of dominant shape.

Proof We know that $u.\xi_{z^+,z} = \xi_z$ for some $u \in S_v(n,r)$, because v is a unit. Since

$$S_v(n,r) \cong S_q(n,r) \oplus v.S_q(n,r)$$

as a $\mathbb{Z}[q,q^{-1}]$ module, we can take u to be in $S_q(n,r)$, by omitting irrelevant terms.

Since $u.\xi_{z^+,z} = u.\xi_{z^+}.\xi_{z^+,z}$, we see that $u.\xi_{z^+}$ is a $\mathbb{Z}[q,q^{-1}]$-linear combination of basis elements of the form ξ_{a,z^+}. It now follows that ξ_z is a $\mathbb{Z}[q,q^{-1}]$-linear combination of codeterminants of shape z^+, i.e. of dominant shape.

Now consider an arbitrary codeterminant, $Y_{i,j}^{\mathrm{wt}(z)}$, where z can be taken to be as in the statement of Lemma 3.1.4 (and so in particular, $z = \ell(z)$). This is equal to $\xi_{i,z}\xi_z\xi_{z,j}$. Using the first part of the proof, we can now rewrite this as

$$\sum_a \mu_a \left(\xi_{i,z}\xi_{a,z^+}\right)\left(\xi_{z^+,z}\xi_{z,j}\right),$$

where the coefficients μ_a lie in $\mathbb{Z}[q,q^{-1}]$. Expanding the parentheses using the product rule, we can finally rewrite the sum as a $\mathbb{Z}[q,q^{-1}]$-linear combination of codeterminants of dominant shape, as required. ∎

3.2. The Quantized Straightening Formula

The next main result will be the proof of the quantum analogue of Woodcock's straightening formula for codeterminants. The aim is to express an arbitrary codeterminant as a linear combination of standard codeterminants. Proposition 3.1.5 shows that it is enough to be able to express an arbitrary codeterminant of dominant shape as a linear combination of standard codeterminants.

Let $\lambda \in \Lambda^+(n,r)$, $\ell = \ell(\lambda)$ and $i \in I'_\lambda \backslash I_\lambda$. Note that ℓ is of parabolic form; denote the associated parabolic subgroup by W_L. We shall construct another element $j \in I(n,r)$ by analysis of the tableau $T = T_i^\lambda$, in exactly the same way as Woodcock [W]. (The symbols i, j and λ will remain fixed throughout the next few lemmas.)

Denote by $[\lambda]$ the Young diagram of λ, considered as a subset of $\mathbb{N}^2$. By the choice of i, T cannot be standard, and so we know there is some $(b,c) \in [\lambda]$ satisfying $T(b,c) \geq T(b+1,c)$. Let (b,c) be lexicographically minimal with respect to this property. Denote by $[\lambda]_a$ the a-th row of $[\lambda]$, and define X_a and Y_a so that $[\lambda]_a = X_a \,\dot{\cup}\, Y_a$ $(a \in \mathbf{n})$, where

$$\begin{array}{lclllcll} X_a &=& [\lambda]_a, & & Y_a &=& \emptyset, & \text{if } a < b; \\ X_b &=& \{\langle b,h\rangle \ : \ 1 \leq h < c\} & & Y_b &=& [\lambda]_b \backslash X_b; & \\ X_a &=& \{x \in [\lambda]_a : T(x) \leq T(y) \text{ for all } y \in Y_{a-1}\}, & & Y_a &=& [\lambda]_a \backslash X_a, & \text{if } a > b. \end{array}$$

Put $\nu_a = |X_a \dot\cup Y_{a-1}|$ for $a \in \mathbf{n+1}$, and define $X_{n+1} = Y_0 = \emptyset$. Let $a_1, \ldots, a_m$ be the elements of $\{a \in \mathbf{n+1} : \nu_a \neq 0\}$, ordered so that $\alpha < \beta \Rightarrow \nu_{a_\alpha} \geq \nu_{a_\beta}$.

Note that $m \leq n$: this is clear if $X_b = \emptyset$. If $X_b \neq \emptyset$, each entry of T in X_{b+1} is at least $b+1$ by the minimality of (b, c). By induction, each of the elements of Y_a is at least $a+1$, so $Y_n = \emptyset$.

We are now ready to define $j \in I$ by the property

$$T_j^\lambda(x) = \alpha \quad \text{if } x \in X_{a_\alpha} \dot\cup Y_{a_\alpha - 1}.$$

Let $\mu = \mathrm{wt}(j)$. By the choice of ordering on the a_α, we have $\mu \in \Lambda^+$, and in fact $\mu \succ \lambda$ since $\nu_\alpha = \lambda_\alpha$ if $\alpha < b$, while $\nu_{b+1} > \lambda_b$.

We also define $j' \in I$ by the property that

$$T_{j'}^\lambda(x) = \alpha \quad \text{if } x \in X_\alpha \dot\cup Y_{\alpha-1}.$$

Note that we may choose the ordering on $a_1, \ldots, a_m$ above in such a way that $j'^+ = j$.

Example Suppose $n = 6$, $r = 18$ and

$$T_i^\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 3 \\ \cline{1-5} 3 & 3 & 3 & 5 \\ \cline{1-4} 5 & 6 \\ \cline{1-2} \end{array}.$$

Then

$$T_j^\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 & 2 \\ \cline{1-5} 2 & 2 & 2 & 4 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array}$$

and

$$T_{j'}^\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 & 3 & 3 \\ \cline{1-5} 3 & 3 & 3 & 4 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array}.$$

The element j is given by

$$\underbrace{1,1,1,1,1,1,1}_{X_1}, \underbrace{3,3}_{X_2}, \underbrace{2,2,2}_{Y_2}, \underbrace{2,2,2}_{X_3}, \underbrace{4}_{Y_3}, \underbrace{4}_{X_4}, \underbrace{5}_{Y_4},$$

where the sections of j corresponding to each nonempty X_a and Y_a have been marked.

Similarly, the element j' is given by

$$\underbrace{1,1,1,1,1,1,1}_{X_1}, \underbrace{2,2}_{X_2}, \underbrace{3,3,3}_{Y_2}, \underbrace{3,3,3}_{X_3}, \underbrace{4}_{Y_3}, \underbrace{4}_{X_4}, \underbrace{5}_{Y_4}.$$

The weights λ and μ are given by

$$\lambda = (7, 5, 4, 2, 0, 0)$$

and

$$\mu = (7, 6, 2, 2, 1, 0).$$

The following lemma is very useful for reducing questions about q-codeterminants to questions about their classical analogues.

Lemma 3.2.1 Suppose $a, b, d, e \in I(n, r)$. Assume that $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_r$ (which implies that $\langle a, b\rangle$ and $\langle b, a\rangle$ are H-relaxed).

(i) Write

$$\xi_{a,b}\xi_{d,e} = \sum_{a' \sim a} c_{a',e}\xi_{a',e}.$$

Then the elements $c_{a',e}$ lie in $\mathbb{N}[q]$.

(ii) Write

$$\xi_{e,d}\xi_{b,a} = \sum_{e' \sim e} c'_{e',a}\xi_{e',a}.$$

Then the elements $c'_{e',a}$ lie in $\mathbb{N}[q]$.

Proof It is clear from the product rule for the q-Schur algebra given in [BLM, Proposition 1.2] that a product of two basis elements is a $\mathbb{Z}[q]$-combination of other basis elements. To prove both parts of the lemma, it therefore suffices to show that the coefficients occurring lie in $\mathbb{N}[v, v^{-1}]$.

We know from [DJ3, Theorem 1.11] that there is an anti-automorphism of the q-Schur algebra which sends an element $\xi_{\alpha,\beta}$ to $\xi_{\beta,\alpha}$. This implies that it is enough to prove (i), because (ii) is an immediate consequence of it.

We now concentrate on (i). We may assume that $\langle d, e\rangle$ is an H-relaxed pair. Define A to be the matrix satisfying $e_A = \xi_{d,e}$ and define s_A and m_A as in §2.2. The parabolic subgroup of W which is the stabiliser of e is denoted by W_e, and we write $\mathcal{D}_{d,e}$ for the set of distinguished right coset representatives corresponding to the parabolic subgroup which is the stabiliser of the pair (d, e).

Any relaxed basis element $u_{\alpha,\beta}$ occurring in the product $\Pi_0 := \xi_{a,b}\xi_{d,e}$ must have $\beta = e$ and $\alpha \sim a$. Now Corollary 2.2.5 implies that the coefficient of $u_{\alpha,\beta}$ in Π_0 is the same as the coefficient in the product $\xi_{a,b}x_{d,e}$, where $x_{d,e} \in T^r(M_n)$ is given by

$$x_{d,e} := v^{s_A - m_A} u_{d,e}. \left(\sum_{w \in W_e \cap \mathcal{D}_{d,e}} T_w \right).$$

Because we assumed that $\langle d, e\rangle$ was H-relaxed, we find by induction on left subwords of w and iteration of equation (1) in §3.0 that if T_w is one of the elements in the sum, then

$$u_{d,e}.T_w = v^* u_{dw,ew},$$

for some integer denoted by $*$. It follows that $x_{d,e}$ is an $\mathbb{N}[v, v^{-1}]$ combination of basis elements of tensor matrix space.

Now consider $e_B = \xi_{a,b}$ as an element of tensor matrix space. We find from Theorem 2.2.7 that

$$\xi_{a,b} = v^{s_B - m_B} u_{a,b}. \left(\sum_{w \in \mathcal{D}_{a,b}} T_w \right).$$

We find that for each w in the sum,

$$u_{a,b}.T_w = v^* u_{aw,bw}.$$

This is proved using the hypotheses on the pair $\langle a, b\rangle$, induction on left subwords of w, and iteration of equation (1) in §3.0. An argument similar to the one in the preceding paragraph shows that $\xi_{a,b}$ is an $\mathbb{N}[v, v^{-1}]$-combination of basis vectors of tensor matrix space.

It is now immediate from Corollary 2.2.5 that the elements $c_{a',e}$ lie in $\mathbb{N}[v, v^{-1}]$, which completes the proof. ∎

Definitions Let Π_0 be the element of $S_q(n, r)$ given by

$$\xi_{i,j'}\xi_{j',l},$$

and let Π_1 be the element of $S(n, r)$ given by the same expression.

Let Π_2 be the element of $S(n, r)$ given by

$$\xi_{i,j}\xi_{j,l}.$$

Lemma 3.2.2 Express Π_0 in terms of the basis elements of the q-Schur algebra $\{\xi_{\alpha,\beta}\}$ (where the pairs $\langle \alpha, \beta\rangle$ are H-relaxed) by writing

$$\Pi_0 = \sum_{i' \in I'_\lambda} c_{i',l}\xi_{i',l},$$

where $c_{i',l} \in \mathbb{Z}[q]$. Then $c_{i,l}$ is a power of q and $i' \succeq i$ whenever $c_{i',l}$ is nonzero.

Proof Note that the product $\Pi_0 = \xi_{i,j'}\xi_{j',l}$ satisfies the hypotheses of Lemma 3.2.1 (ii). Write $c'_{i',l}$ for the integer which is the specialisation as q is replaced by 1 of $c_{i',l}$. It is now enough in the light of Lemma 3.2.1 to prove that the coefficient $c'_{i,l}$ in Π_1 is 1 and that $i' \succeq i$ whenever $c'_{i',l} \neq 0$, because $c'_{i,l}$ is equal to 1 if and only if $c_{i,l}$ is a (nonnegative) power of q, and $c'_{i',l}$ is nonzero if and only if $c_{i',l}$ is nonzero. (This is true because the coefficients in the statement of Lemma 3.2.1 all lie in $\mathbb{N}[q]$.)

It is immediate from Schur's product rule and the construction of j and j' that Π_1 and Π_2 are equal. The element Π_2 was studied in [W], where it was proved that the coefficient of $\xi_{i,l}$ in Π_2 was equal to 1 and that any basis element $\xi_{i',l}$ appearing in Π_2 satisfied $i' \succeq i$.

The proof now follows. ∎

Lemma 3.2.3 The element Π_0 can be written as an $\mathcal{A}$-linear sum of q-codeterminants of shape μ.

Proof We see from Lemma 3.1.4 (i) that

$$\tau_{j',j}\tau_{j,j'} = v^* \tau_{j',j'}$$

for some integer denoted by $*$. It follows that we can write

$$\Pi_0 = \xi_{i,j'}\xi_{j',l} = \xi_{i,j'}\xi_{j',j'}\xi_{j',l} = v^* \left(\xi_{i,j'}\tau_{j',j}\right)\left(\tau_{j,j'}\xi_{j',l}\right).$$

Now we expand each pair of parentheses in terms of the basis $\xi_{\alpha,\beta}$ to obtain an $\mathcal{A}$-linear sum of q-codeterminants of shape $\mu = \mathrm{wt}(j)$. ∎

Lemma 3.2.4 Let $\lambda \in \Lambda^+(n,r)$, $\ell = \ell(\lambda)$, and $i \in I'_\lambda \backslash I_\lambda$. Then there exist elements $c_{i',\ell} \in \mathcal{A}$ and $j \in I(n,r)$ such that $\mu = \mathrm{wt}(j) \in \Lambda^+$, $\mu \succ \lambda$, and

$$\xi_{i,\ell} = q^*\Pi_0 + \sum_{\substack{i' \in I'_\lambda \\ i' \succ i}} c_{i',\ell}\xi_{i',\ell}.$$

Thus

$$\xi_{i,\ell} = \sum_{\substack{a \sim i \\ b \sim \ell}} c'_{a,b} Y^\mu_{a,b} + \sum_{\substack{i' \in I'_\lambda \\ i' \succ i}} c_{i',\ell}\xi_{i',\ell},$$

where the scalars c and c' lie in $\mathcal{A}$.

Proof The elements j and μ are constructed from i and λ as previously, where it was also shown that $\mu \in \Lambda^+$ and $\mu \succ \lambda$. The first assertion follows from rearranging the expression for Π_0 given by

Lemma 3.2.2, and using the fact that the coefficient of $\xi_{i,\ell}$ is a power of q (i.e. a unit). The second assertion follows from the first by re-expressing Π_0 as an $\mathcal{A}$-linear sum of codeterminants of shape μ, as explained in Lemma 3.2.3.

∎

We require the following well-known property of q-Schur algebras in the proof of the main result.

Proposition 3.2.5 There is an anti-isomorphism $* : S_q(n,r) \to S_q(n,r)$ given by either

$$(\phi^d_{\lambda,\mu})^* = \phi^{d^{-1}}_{\mu,\lambda},$$

$$(e_A)^* = e_{A^\top},$$

$$\text{or} \quad (\xi_{a,b})^* = \xi_{b,a}.$$

Thus $(Y^\nu_{a,b})^* = Y^\nu_{b,a}$.

Proof For the proof of this with respect to the ϕ, see [DJ3, Theorem 1.11]. The other two equivalent definitions follow from the equivalence of the three bases.

The last assertion is immediate from the definition of codeterminants. ∎

Definition Following [W], we define

$$\mathcal{B} := \{(\lambda, i, j) : \lambda \in \Lambda^+, i, j \in I'_\lambda\},$$

and we partially order $\mathcal{B}$ by $(\lambda, i, j) \preceq (\mu, i', j')$ if and only if $\lambda \prec \mu$ or $(\lambda = \mu,\ i \preceq i'$ and $j \preceq j')$.

We can now state and prove the main theorem.

Theorem 3.2.6 (the Straightening Formula for Quantized Codeterminants)

Let $Y^\lambda_{i,j}$ be any codeterminant. Then $Y^\lambda_{i,j}$ is a $\mathbb{Z}[q,q^{-1}]$-linear combination of standard codeterminants.

Proof By Proposition 3.1.5, it is enough to prove the theorem in the case where $\lambda \in \Lambda^+$. Since

$$S_v(n,r) \cong S_q(n,r) \oplus v.S_q(n,r)$$

as a $\mathbb{Z}[q,q^{-1}]$ module, and all codeterminants $Y^\mu_{a,b}$ lie in $S_q(n,r)$, we see that it is enough to prove the result that $Y^\lambda_{i,j}$ is an $\mathcal{A}$-linear combination of standard codeterminants.

Suppose $Y^\lambda_{i,j}$ is not standard. It is enough to show that we can write it as an $\mathcal{A}$-linear combination of codeterminants corresponding to elements of $\mathcal{B}$ which are strictly higher in the partial order on $\mathcal{B}$, and then use induction on the poset $(\mathcal{B}, \preceq)$.

Assume first that $i \notin I_\ell$. Construct the elements $j \in I(n,r)$ and $\mu \in \Lambda^+$ as in Lemma 3.2.4, and set $\ell = \ell(\lambda)$.

By definition, $Y^\lambda_{i,j} = \xi_{i,\ell} \times \xi_{\ell,j}$. We now substitute for $\xi_{i,\ell}$ using the formula in Lemma 3.2.4 to obtain

$$Y^\lambda_{i,j} = \left(\sum_{\substack{a\sim i \\ b\sim \ell}} c'_{a,b} Y^\mu_{a,b} \times \xi_{\ell,j} \right) + \left(\sum_{\substack{i' \in I'_\lambda \\ i' \succ i}} c_{i',\ell} \xi_{i',\ell} \xi_{\ell,j} \right),$$

where the scalars c and c' all lie in $\mathcal{A}$. Next, we can rewrite the expression in the left-hand set of parentheses using the fact that

$$Y^\mu_{a,b} \xi_{\ell,j} = \xi_{a,\ell(\mu)} (\xi_{\ell(\mu),b} \xi_{\ell,j}),$$

and then expanding the product $\xi_{\ell(\mu),b}\xi_{\ell,j}$ as a sum of basis elements $\xi_{\ell(\mu),d}$ by using the product rule. Finally we can write the product $Y^\mu_{a,b}\xi_{\ell,j}$ as an $\mathcal{A}$-linear sum of codeterminants $Y^\mu_{a,d}$. This gives

$$Y^\lambda_{i,j} = \left(\sum_{\substack{a\sim i \\ d\sim j}} c''_{a,d} Y^\mu_{a,d} \right) + \left(\sum_{\substack{i' \in I'_\lambda \\ i' \succ i}} c_{i',\ell} Y^\ell_{i',j} \right),$$

where the scalars c'' and c all lie in $\mathcal{A}$.

Note that we have now succeeded in expressing $Y^\lambda_{i,j}$ as the sum of codeterminants corresponding to strictly higher elements of $\mathcal{B}$, as required.

The other case, where $i \in I_\lambda$ but $j \notin I_\lambda$ follows by symmetry by using Proposition 3.2.5 followed by the preceding argument, then Proposition 3.2.5 again. This completes the proof. ∎

Corollary 3.2.7 The set of standard codeterminants forms a free $\mathbb{Z}[q,q^{-1}]$-basis of $S_q(n,r)$.

Proof The cardinality of this set is $\binom{n^2+r-1}{r}$, because it is the same as the cardinality of the corresponding set in the classical case. (See [G1, §2] for a proof of the latter.) This is the same as the dimension of $S_q(n,r)$. It is also known (see [DJ3, Theorem 1.4]) that the elements e_A form a free basis for $S_q(n,r)$. It is therefore enough to show that each e_A is a $\mathbb{Z}[q,q^{-1}]$-linear combination of standard codeterminants. Theorem 3.2.6 shows that it is enough to prove that each basis element $\xi_{i,j}$ is equal to a codeterminant. This is clear, because $\xi_{i,j} = \xi_{i,j}\xi_j$. ∎

We end this chapter by giving a brief summary of the steps needed to straighten an arbitrary codeterminant, $Y^{\lambda'}_{i',k'}$.

(1) Find the element $z \in I(n,r)$ satisfying $a < b \Rightarrow z_a \leq z_b$ and $\mathrm{wt}(z) = \lambda'$. Use the identity $\tau_{z,z^+}\tau_{z^+,z} = v^*\tau_{z,z}$ of Lemma 3.1.4 (i) and the techniques of Proposition 3.1.5 to express the codeterminant as a sum of codeterminants of dominant shape.

(2) Straighten each non-standard codeterminant $Y^\lambda_{i,k}$ arising from the previous procedure. If $i \in I_\lambda$ but $k \notin I_\lambda$, straighten the codeterminant $Y^\lambda_{k,i}$ instead and use Proposition 3.2.5 to "turn the results around" at the end.

(3) We may now concentrate on straightening codeterminants of form $Y^\lambda_{i,k}$, where λ is a dominant weight and $i \notin I_\lambda$. Construct the elements j and j' from i and λ, as described at the beginning of §3.2. Using the formula for $\xi_{i,\ell}$ given in Lemma 3.2.4, and the techniques of the proof of Theorem 3.2.6, rewrite $Y^\lambda_{i,k}$ as a sum of codeterminants corresponding to strictly higher elements of the poset $\mathcal{B}$.

(4) Iterate parts (2) and (3) until all codeterminants appearing in the sum are standard.

4. q-Schur Algebras as Quotients of Quantized Enveloping Algebras

The main aim of this chapter is to investigate the properties of the surjective algebra homomorphism $\theta = \theta_r$ from the quantized enveloping algebra to the q-Schur algebra. In particular, we would like to be able to find preimages of elements of the q-Schur algebra under this map, and we would like a description of $\ker \theta$. These problems are solved at the end of this chapter, and are related to the problem of straightening q-codeterminants which was solved in §3.

4.0 Conventions and review of §3

The algebras U, U^-, U^0 and U^+ used in this chapter refer to the $\mathcal{A}$-algebras $U_{\mathcal{A}}(gl_n)$, $U_{\mathcal{A}}^-(gl_n)$, $U_{\mathcal{A}}^0(gl_n)$ and $U_{\mathcal{A}}^+(gl_n)$. These were defined in §1.1.1. As usual, the q-Schur algebra $S_q(n, r)$ is defined over $\mathbb{Z}[q, q^{-1}]$ and the v-Schur algebra $S_v(n, r)$ is defined over $\mathcal{A}$.

It is worth re-stating the relationship between tableaux and q-codeterminants in a way compatible with the matrix basis $\{e_A : A \in \Theta_r\}$.

Consider any ordered pair $\langle T, T' \rangle$ of standard tableaux of the same shape, each consisting of r boxes and having entries in $\mathbf{n}$. We showed in §1.4.2 how to associate to $\langle T, T' \rangle$ a certain q-codeterminant, $e_A e_{A'}$. An equivalent way of defining this is as follows.

The entry A_{ij} of A is defined to be the number of occurrences of i in the j-th row of the tableau T, or zero if there is no such entry.

The entry A'_{ij} of A' is defined to be the number of occurrences of j in the i-th row of the tableau T', or zero if there is no such entry.

The fact that the tableaux are of the same shape forces the product $e_A e_{A'}$ to be nonzero, because it is of form $\xi_{i,\ell}\xi_{\ell,j}$, and by using the product rule for $S_q(n, r)$, we find that this is nonzero, and hence $e_A e_{A'}$ is a codeterminant. We call such a codeterminant a *standard codeterminant*. This definition agrees with that in §3.

It should be noted that, because the tableaux T and T' are standard, the matrix A must be lower triangular, and the matrix A' must be upper triangular.

Example

Suppose $n = 3$, $r = 6$,

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ ,$$

and

$$T' = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ .$$

Then the matrices A and A' are given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$A' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rest of this chapter is divided into two sections. In §4.1, we study the restriction of θ to certain subalgebras of U. In §4.2, we use the properties of quantized codeterminants to complete our description of the relationship between U and $S_q(n, r)$.

4.1. The restriction of θ to U^-, U^0 and U^+

The next aim is to obtain precise and general descriptions for $U^+ \cap \ker\theta$ and for $U^- \cap \ker\theta$, concentrating on the first case, because the second case is essentially similar.

Definition Let X be the matrix

$$\begin{pmatrix} 0 & c_1 & c_2 & c_3 & \cdots & c_{n-1} \\ 0 & 0 & c_n & c_{n+1} & \cdots & c_{2n-3} \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & c_N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We define

$$y_{X,r} := \sum_{\lambda_1 + \cdots + \lambda_n + |X| = r} [(X, \lambda_1, \ldots, \lambda_n)],$$

summed over all sets of nonnegative integers $\lambda_1, \ldots, \lambda_n$, where the matrix $(X, \lambda_1, \ldots, \lambda_n)$ is given by $X + \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $|X|$ denotes the sum of the entries in X.

It is known that, for $1 \le i \le n-1$, E_i maps under θ to $y_{X,r}$, where X is the matrix $E_{i,i+1}$, having 1 in $(i, i+1)$ place and zeros elsewhere.

In order to prove the main result of this section, we will rely on the the following lemma, which is a simple corollary of a lemma by Beilinson, Lusztig and MacPherson.

Lemma 4.1.1 Let X be a strictly upper triangular $n \times n$ matrix, with $|X| \leq r$. Then $\theta(E_h) \times y_{X,r}$ is given by

$$(1-\delta_{|X|,r})v^{\beta(h+1)}[X_{h,h+1}+1]y_{X+E_{h,h+1},r} + \sum_{p\in[h+2,n];X_{h+1,p}\geq 1} v^{\beta(p)}[X_{h,p}+1]y_{X+E_{h,p}-E_{h+1,p},r}$$

where, as usual, $E_{i,j}$ is the matrix with 1 in the (i,j) place, and 0 everywhere else.

Here,

$$\beta(p) := \sum_{j>p}(X_{h,j} - X_{h+1,j}).$$

Let X be a strictly lower triangular $n \times n$ matrix. Then $\theta(F_h) \times y_{X,r}$ is given by

$$(1-\delta_{|X|,r})v^{\beta'(h)}[X_{h+1,h}+1]y_{X+E_{h+1,h},r} + \sum_{p\in[1,h-1];X_{h,p}\geq 1} v^{\beta'(p)}[X_{h+1,p}+1]y_{X-E_{h,p}+E_{h+1,p},r}.$$

Here,

$$\beta'(p) := \sum_{j<p}(X_{h+1,j} - X_{h,j}).$$

Proof This follows from [BLM, Lemma 3.4 (a2)], using the formula for $\theta(E_i)$. ∎

Definition

Let X be an $n \times n$ matrix with entries in $\mathbb{Z}$. We say X has the property $P(i,j)$ (where i and j are integers satisfying $1 \leq i \leq j < n$) if it satisfies one of two conditions. If row $j+1$ of X consists entirely of zeros, then X has the property $P(i,j)$. Otherwise, let p be minimal such that $X_{j+1,p}$ is nonzero. If for each integer s such that $i < s < j+1$, $X_{s,t} = 0$ for $t \geq s+p-j$, then X has the property $P(i,j)$. If neither condition holds, then X does not have the property $P(i,j)$.

Example

Suppose

$$X = \begin{pmatrix} 3 & 5 & 1 & 0 & 3 & 4 \\ 5 & 2 & 3 & 0 & 3 & 1 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 3 & 0 & 2 & 4 & 4 \end{pmatrix}.$$

Then X has the properties $P(k,k)$ for all $1 \leq k < 6$, and also $P(1,3)$, $P(2,3)$, $P(2,4)$ and $P(3,4)$.

Lemma 4.1.2 Let $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ be a positive root.

Assume X is a strictly upper triangular matrix such that if D is a diagonal n by n matrix with nonnegative integer entries, then $X + D$ has the property $P(i,s)$ for all $i < s \leq j$. Then $\theta(E_\alpha) \times y_{X,r}$ is given by

$$(1 - \delta_{|X|,r}) v^{x(j+1)} [X_{i,j+1} + 1] y_{X+E_{i,j+1},r} + \sum_{p \in [j+2,n]; X_{j+1,p} \geq 1} v^{x(p)} [X_{i,p} + 1] y_{X+E_{i,p}-E_{j+1,p},r}.$$

Here,

$$x(p) := \sum_{m>p} (X_{i,m} - X_{j+1,m}).$$

Proof The proof is by induction on $n' = j - i$. The case $n' = 0$ is done by Lemma 4.1.1. It now suffices to check that, when $i < j$ and $\beta = \alpha - \alpha_j$, if the hypothesis works for E_β and for E_j (by induction), then it works for $E_\alpha := E_\beta E_j - v^{-1} E_j E_\beta$.

Using the inductive hypothesis applied to E_β (since we know any matrix $X + D$ has the property $P(i,s)$ for all $i < s \leq j-1$), we can assume that

$$\theta(E_\beta) \times y_{X,r} = (1 - \delta_{|X|,r}) v^{x'(j)} [X_{i,j} + 1] y_{X+E_{i,j},r} + \sum_{p \in [j+1,n]; X_{j,p} \geq 1} v^{x'(p)} [X_{i,p} + 1] y_{X+E_{i,p}-E_{j,p},r},$$

where

$$x'(p) := \sum_{m>p} (X_{i,m} - X_{j,m}).$$

Rephrasing this informally, and concentrating on one particular term, $[Y]$, of $y_{X,r}$, we find that a typical term in the action of E_β on $[Y]$ is to decrease an entry in the (j,p)-place of Y by 1, for some suitable p, to increase the entry in the (i,p) place by 1 (resulting in a new matrix Y') and to multiply by

$$v^{y'(p)} [Y'_{i,p}],$$

where

$$y'(p) := \sum_{m>p} (Y_{i,m} - Y_{j,m}).$$

Similarly, considering the action of E_j on $[Z]$ we find that we find that the action of a typical term is, for some suitable p', to decrease an entry in the $(j+1,p')$-place of Y by 1, to increase the entry in the (j,p') place by 1 (resulting in a new matrix Z') and to multiply by

$$v^{z'(p')} [Z'_{i,p'}],$$

where

$$z'(p') := \sum_{m>p'} (Z'_{j,m} - Z'_{j+1,m}).$$

The crucial issue is the extent to which these two actions fail to commute if $p \neq p'$. Let us suppose that $p \neq p'$. We claim that the property P (as above) forces $p \leq p'$. Suppose to the contrary that $p > p'$. This implies that the matrix Y has nonzero entries in the $(j+1, p')$ place and in the (j, p) place. However, the matrix Y is assumed to have property $P(i, j)$, and we are also assuming that row $j+1$ is not empty. Let p'' be minimal such that $Y_{j+1,p''}$ is nonzero. Then we must have $p'' \leq p' < p$. Putting $s = j$ in the definition of the property P (which is valid since $n' = j - i > 0$), we see that $Y_{j,t} = 0$ for $t \geq p''$, so in particular, $Y_{j,p} = 0$. This is a contradiction.

Next, consider a typical term, $[A]$, of $y_{X,r}$. Of course, A is an upper triangular matrix. Acting $E_\beta E_j$ on this matrix, on the left, we obtain the new matrix $[A']$ (with entries (j, p) and $(j+1, p')$ decreased by 1, and entries (i, p) and (j, p') increased by 1) with coefficient

$$[A_{i,p} + 1][A_{j,p'} + 1]v^x,$$

where

$$x = \sum_{m > p'} (A_{j,m} - A_{j+1,m}) + \sum_{m > p} (A'_{i,m} - A'_{j,m}),$$

i.e.

$$\sum_{m > p'} (A_{j,m} - A_{j+1,m}) + \sum_{m > p} (A_{i,m} - A_{j,m}) - 1.$$

Similarly, computing the action of $E_j E_\beta$, we find that matrix $[A']$ occurs with coefficient

$$[A_{i,p} + 1][A_{j,p'} + 1]v^{x'},$$

where

$$x' = \sum_{m > p'} (A_{j,m} - A_{j+1,m}) + \sum_{m > p} (A_{i,m} - A_{j,m}),$$

which is exactly v times the coefficient with the elements acting the other way round. Thus, in the action of $E_\beta E_j - v^{-1} E_j E_\beta$, the above terms cancel out.

The only remaining terms correspond to the situation $p = p'$. Using similar methods to the above, we find that the coefficient of $[A']$ in the action of $E_\beta E_j - v^{-1} E_j E_\beta$ on $[A]$ is

$$[A_{i,p} + 1][A_{j,p} + 1]v^{x''},$$

where

$$x'' = \sum_{m > p} (A_{i,m} - A_{j+1,m}).$$

In fact, $A_{j,p}$ is equal to zero, because the matrix $[A]$ has the property $P(i, j)$ and $A_{j+1,p}$ is nonzero. Hence $[A_{j,p} + 1] = 1$.

Notice that this argument is essentially independent of the entries on the diagonal of A. It therefore applies equally well to the action of E_α on $y_{X,r}$, thus establishing the inductive hypothesis for E_α.

This completes the proof. ∎

Recall from §1.1.2 the ordering defined on Φ^+ associated with the elements E_α. Denote the last root in this list by β_1, the second from last by β_2, etc.

Proposition 4.1.3 The element

$$E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)}$$

maps under θ to $y_{X,r}$. If $|X| > r$, then the given basis element maps to zero.

Proof The proof will be by induction on $n' = \sum_{k=1}^N c_k$. The case $n' = 1$ is done by Lemma 4.1.2. For the general case, we will prove the equivalent statement that

$$\theta(E_{\beta_N}^{c_N} \cdots E_{\beta_1}^{c_1}) = \left(\prod_{k=1}^N [c_k]!\right) y_{X,r}$$

Let p be maximal such that c_p is nonzero, and let X' be the matrix obtained by decreasing the entry in the position corresponding to c_p in X by 1. We now need to check that E_{β_p} acts on $y_{X',r}$ to give $[c_p]y_{X,r}$, as expected. If $n' > r+1$ then there is nothing to prove, because $y_{X',r} = 0$, by the inductive hypothesis. Otherwise, express β_p in the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. The ordering chosen for the positive roots guarantees that if D is a diagonal n by n matrix with nonnegative integer entries, then the matrix $X' + D$ has no nonzero entries between rows $i+1$ and $j+1$ inclusive, except possibly on the diagonal. This means that $X' + D$ has the properties $P(i,s)$ for $i < s \leq j$, so we can apply Lemma 4.1.2. Since $X_{j+1,t} = 0$ for $t > j+1$, we find that only the first term occuring in Lemma 4.1.2 can appear. If $n' = r$, then we find that this term too is zero, making our element map under θ to zero, as expected. Further scrutiny of the matrices X' and X reveals that $X_{i,m} = 0$ for $m > j+1$; again this is by properties of the ordering chosen on the positive roots. This means that the quantity $x(j+1)$ occurring in Lemma 4.1.2 is equal to 0, and the inductive step immediately follows. ∎

Corollary 4.1.4 $U^+ \cap \ker\theta$ is generated by those basis elements which map to zero under θ, i.e.

$$U^+ \cap \ker\theta = \left\langle \left\{ E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)} : \sum_{i=1}^N c_i \geq r+1 \right\} \right\rangle.$$

Proof The basis elements which do not map to zero all map to elements with different associated matrices X, and hence their images are linearly independent. The corollary now follows. ∎

Corollary 4.1.5 $\theta(U^+)$ is the subalgebra of $S_q(n,r)$ generated by $y_{X,r}$ for all possible X of zero triangular form.

Proof Clearly $\theta(U^+)$ is the subspace generated by the images of the PBW-type basis elements of U^+. These elements either map to zero, or to a multiple of one of the above elements $y_{X,r}$. ∎

Corollary 4.1.6 The dimension of $\theta(U^+)$ is given by

$$\binom{\binom{n}{2}+r}{r}.$$

Proof Firstly, observe that by elementary properties of Pascal's triangle,

$$\sum_{i=0}^{r}\binom{\binom{n}{2}+i-1}{i}=\binom{\binom{n}{2}+r}{r}.$$

Secondly, recall that the number of compositions of r into m pieces is known to be

$$\binom{m+r-1}{r}.$$

To count the dimension of $\theta(U^+)$, it suffices to enumerate the possible matrices X, because the corresponding $y_{X,r}$ are linearly independent. The possibilities for X correspond to compositions of integers between 0 and r into $n(n-1)/2$ pieces. The result now follows. ∎

Remarks on the behaviour of U^- The corresponding results for U^- are similar in spirit, and correspond to "rotating the basis matrices by a half turn".

Recall the ordering on Φ^+ associated with the elements F_α. Denote the last root in this list by γ_1, the second from last by γ_2, etc. We have the following result.

Proposition 4.1.7

Let X be the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ c_N & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \\ c_{2n-3} & \cdots & c_{n+1} & c_n & 0 & 0 \\ c_{n-1} & \cdots & c_3 & c_2 & c_1 & 0 \end{pmatrix}.$$

We define

$$z_{X,r} := \sum_{\lambda_1+\cdots+\lambda_n+|X|=r} [(X,\lambda_1,\ldots,\lambda_n)],$$

summed over all sets of nonnegative integers $\lambda_1, \ldots, \lambda_n$, where the matrix $(X, \lambda_1, \ldots, \lambda_n)$ is given by $X + \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $|X|$ denotes the sum of the entries in X.

Then the element

$$F_{\gamma_N}^{(c_N)} \cdots F_{\gamma_1}^{(c_1)}$$

maps under θ to $z_{X,r}$. If $|X| > r$, then the given basis element maps to zero.

The intersection of U^- with the kernel of θ is given by

$$U^- \cap \ker\theta = \left\langle \left\{ F_{\gamma_N}^{(a_N)} \cdots F_{\gamma_1}^{(a_1)} : \sum_{i=1}^{N} a_i \geq r+1 \right\} \right\rangle.$$

Proof This is the same as the proof of the corresponding result for U^+, with trivial changes. ∎

We now investigate the case of $U^0 \cap \ker\theta$. To do this, we require the following result.

Proposition 4.1.8

A basis for U^0 is given by the set

$$K_1^{\delta_1} K_2^{\delta_2} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix},$$

where $\delta_i \in \{0, 1\}$ and $t_i \in \mathbb{N}$, and where

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} := \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

Proof This follows from [L2, Proposition 2.14, Theorem 4.5]. ∎

We wish to investigate the images under θ of these basis elements. We know that

$$\theta(K_i) = \sum_{D \in \mathbb{D}_r} v^{d_i} [D],$$

where $D = \mathrm{diag}(d_1, \ldots, d_n)$, $\mathbb{D}_r$ is the set of $n \times n$ diagonal matrices with nonnegative integer entries summing to r, and as usual

$$[D] = v^{-\dim \mathcal{O}_D + \dim pr_1(\mathcal{O}_D)} e_D,$$

but since D is diagonal, it is not hard to see that $[D] = e_D$. Since e_D corresponds to $\phi^1_{\lambda\lambda}$ in the notation of Dipper and James [DJ3], we see that the e_D form a set of mutually orthogonal idempotents. Armed with this information, it is easy to work out the images of products of the K_i, because the multiplication in the v-Schur algebra is componentwise.

Lemma 4.1.9

$$\theta\left(K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}\right)=\sum_{D\in\mathbb{D}_r}v^{\delta_1d_1+\cdots+\delta_nd_n}\begin{bmatrix}d_1\\t_1\end{bmatrix}\begin{bmatrix}d_2\\t_2\end{bmatrix}\cdots\begin{bmatrix}d_n\\t_n\end{bmatrix}[D].$$

Proof Because of the preceding discussion, it is enough to verify the Lemma for

$$\theta\left(\begin{bmatrix}K_i;0\\t_i\end{bmatrix}\right).$$

Recalling the definition of

$$\begin{bmatrix}K_i;0\\t_i\end{bmatrix},$$

we find that it is important to find the images under θ of elements of the form

$$\frac{K_iv^{-s+1}-K_i^{-1}v^{-s-1}}{v^s-v^{-s}},$$

which, since θ is linear, can easily be seen to be

$$\sum_{D\in\mathbb{D}_r}\frac{v^{d_i-s+1}-v^{-d_i+s-1}}{v^s-v^{-s}}[D],$$

in other words

$$\sum_{D\in\mathbb{D}_r}\frac{[d_i-s+1]}{[s]}[D].$$

It now follows that

$$\theta\left(\begin{bmatrix}K_i;0\\t_i\end{bmatrix}\right)=\sum_{D\in\mathbb{D}_r}\begin{bmatrix}d_i\\t_i\end{bmatrix}[D].$$

The Lemma now follows easily. ∎

Corollary 4.1.10

a) If $\sum_{i=1}^n t_i=r$, then

$$\theta\left(K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}\right)=v^{\delta_1t_1+\cdots+\delta_nt_n}[T],$$

where $T\in\mathbb{D}_r$ and the diagonal entries of T are $t_1,\ldots,t_n$.

b) If $\sum_{i=1}^n t_i>r$, then

$$\theta\left(K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}\right)=0.$$

Proof Note that if $d_i<t_i$ then

$$\begin{bmatrix}d_i\\t_i\end{bmatrix}=0.$$

It now follows, in case a), that there is only one term in the sum given in Lemma 4.1.9, namely the one given. By a similar argument, we find that in case b) there are no surviving terms in the sum of Lemma 4.1.9. ∎

Corollary 4.1.10 provides us with a large part of $U^0 \cap \ker\theta$. The rest comes from considering basis elements B of U^0 satisfying $\sum_{i=1}^n t_i \leq r$, and subtracting off other elements corresponding, via Corollary 4.1.10, to the terms appearing in $\theta(B)$, yielding:

Corollary 4.1.11

$$\theta\left(K_1^{\delta_1}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix} - \sum_{D\in\mathbb{D}_r} v^{\delta_1 d_1+\cdots+\delta_n d_n}\begin{bmatrix}d_1\\t_1\end{bmatrix}\cdots\begin{bmatrix}d_n\\t_n\end{bmatrix}\begin{bmatrix}K_1;0\\d_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\d_n\end{bmatrix}\right) = 0$$

Proof Use Lemma 4.1.9 and Corollary 4.1.10. ∎

Using the basis of U^0 described earlier, it is now possible to describe $U^0 \cap \ker\theta$ exactly.

Proposition 4.1.12 Let $\kappa_{n,r}$ be the function on the basis of U^0 given in Proposition 4.1.8 sending

$$K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}$$

to

$$K_1^{\delta_1}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix} - \sum_{D\in\mathbb{D}_r} v^{\delta_1 d_1+\cdots+\delta_n d_n}\begin{bmatrix}d_1\\t_1\end{bmatrix}\cdots\begin{bmatrix}d_n\\t_n\end{bmatrix}\begin{bmatrix}K_1;0\\d_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\d_n\end{bmatrix}.$$

Then a basis for $U^0 \cap \ker\theta$ is given by $\{\kappa_{n,r}(B)\}$ as B runs through basis elements of the form

$$K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}$$

except those where all the δ_i are equal to zero and $\sum_{i=1}^n t_i = r$.

Proof By Corollary 4.1.11, $\theta(\kappa_{n,r}(B)) = 0$ for all basis elements B. By Corollary 4.1.10, if $\sum_{i=1}^n t_i \neq r$, or $\sum_{i=1}^n t_i = r$ and not all the δ_i are zero, then B occurs as a "lowest term" (since $d_i \geq t_i$) of coefficient 1 in $\kappa_{n,r}(B)$. In this case, $\kappa_{n,r}(B)$ is nonzero. This proves that the elements given in the statement of the Proposition are linearly independent in U^0.

Also by Corollary 4.1.10, if $\sum_{i=1}^n t_i = r$, and all the δ_i are zero, then $\kappa_{n,r}(B) = 0$. In this case, $\theta(B)$ is some diagonal basis element $[D]$, and all such basis elements turn up as the image of exactly one such B. This proves that that the elements $\{\kappa_{n,r}(B)\}$ as given in the statement of the proposition span $U_0 \cap \ker\theta$, as required. ∎

Definition Take as a basis B^0 of U^0 the elements

$$K_1^{\delta_1} K_2^{\delta_2} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix},$$

where $t_i \in \mathbb{N}_0$, and $\sum_{i=1}^n t_i > r$ or both $\sum_{i=1}^n t_i = r$ and all the δ_i are equal to zero, together with the elements

$$K_1^{\delta_1} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix} - \sum_{D \in \mathbb{D}_r} v^{\delta_1 d_1 + \cdots + \delta_n d_n} \begin{bmatrix} d_1 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} d_n \\ t_n \end{bmatrix} \begin{bmatrix} K_1; 0 \\ d_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ d_n \end{bmatrix},$$

where $t_i \in \mathbb{N}_0$, and $\sum_{i=1}^n t_i < r$ or both $\sum_{i=1}^n t_i = r$ and at least one δ_i is nonzero.

This basis will be useful in §4.2.

Note that we now know that $B = B^- \otimes B^0 \otimes B^+$ is a basis for U. This provides a very convenient context in which to study θ.

4.2. Codeterminants, explicit surjectivity and ker (θ)

We now study some of the properties of $U^{\geq 0}$, the subalgebra of U generated by U^0 and U^+, and $U^{\leq 0}$, the subalgebra of U generated by U^0 and U^-. We concentrate on $U^{\geq 0}$.

Proposition 4.2.1 Let A be an upper triangular matrix corresponding to a basis element of $S_q(n,r)$. Let t_i be the sum of the entries in row i of A. Then the element of $U^{\geq 0}$ given by

$$\begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix} E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)}$$

maps under θ to $[A]$, where the c_i correspond to the entries above the diagonal in A according to the order on the positive roots associated with the elements E_α.

Proof This follows from Proposition 4.1.3, Corollary 4.1.10 and the following fact. In the q-Schur algebra as presented by Dipper and James [DJ3, §2], they show that that

$$\phi_{ab}^d \phi_{cc}^1 = \delta_{bc} \phi_{ac}^d,$$

and that

$$\phi_{aa}^1 \phi_{bc}^d = \delta_{ab} \phi_{ac}^d.$$

Recalling the discussion before Lemma 4.1.9, and the correspondence between $[A]$-basis and $\phi_{\lambda\mu}$-basis, it becomes clear that $[D].[A] = [A]$ if the diagonal entries of $[D]$ are the row sums of $[A]$, and $[D][A] = 0$ otherwise. Therefore, the effect of the U^0 part of the element given in the Proposition is to pick out one term of the $y_{X,r}$ expression corresponding to the U^+-part of the element in the

Proposition. The matrix B corresponding to the term $[B]$ picked out in this way has the same row sums as A, and is the same above the diagonal, so it must actually be A.

These remarks suffice to prove the Proposition. ∎

Corollary 4.2.2 The image of $U^{\geq 0}$ under θ is precisely the subspace of $S_q(n,r)$ spanned by the upper triangular basis elements.

Proof All basis elements in the basis $B^0 \otimes B^+$ of $U^{\geq 0}$ map under θ to sums of upper triangular basis elements. We know from Proposition 4.2.1 that we can find an element of $U^{\geq 0}$ mapping to any desired upper triangular basis element. This completes the proof. ∎

Remark The image of $U^{\geq 0}$ under θ is, in the classical case, precisely the so-called *Borel subalgebra* of the Schur algebra, often denoted by S^+. Similarly, the image of $U^{\leq 0}$ is the subalgebra denoted by S^- in the literature, and $\theta(U^{\leq 0})$ is the subalgebra spanned by lower triangular basis elements.

The situation for $U^{\leq 0}$ is extremely similar. For example, Proposition 4.2.1 becomes:

Let A be a lower triangular matrix corresponding to a basis element of $S_q(n,r)$. Let t_i be the sum of the entries in column i of A. Then the element of $U^{\geq 0}$ given by

$$F_{\gamma_N}^{(c_N)} \cdots F_{\gamma_1}^{(c_1)} \begin{bmatrix} K_1;0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2;0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n;0 \\ t_n \end{bmatrix}$$

maps under θ to $[A]$, where the c_i correspond to the entries below the diagonal in A according to the usual order on the positive roots associated with the elements F_α.

We are now in a position to describe the relationship between $U(gl_n)$ and $S_v(n,r)$. It is known (see [L3, §3.2]) that

$$U \cong U^- \otimes U^0 \otimes U^+.$$

We will work with the basis $B = B^- \otimes B^0 \otimes B^+$ for the algebra U.

We find from earlier results that all but finitely many elements of B map to zero under θ. We will show that the elements of B which do not map to zero map to elements of $S_v(n,r)$ which are closely related to codeterminants.

Definition If $Y_{i,j}^\lambda$ is a typical q-codeterminant given by the factorisation $e_A e_{A'}$, then denote by $\widehat{Y}_{i,j}^\lambda$ the v-codeterminant given by $[A][A']$. Recall that the triple (λ, i, j) is related to the product

$[A][A']$ as follows. The entry $A_{a,b}$ of A is defined to be the number of occurrences of a in the b-th row of the tableau of shape λ corresponding to i. The entry $A'_{a,b}$ of A' is defined to be the number of occurrences of b in the a-th row of the tableau of shape λ corresponding to j.

We now recall the straightening formula for q-codeterminants.

Proposition 4.2.3

Let $Y^{\lambda}_{i,j}$ be any q-codeterminant.

Then $Y^{\lambda}_{i,j}$ is a $\mathbb{Z}[q, q^{-1}]$-linear combination of standard q-codeterminants. The coefficients arising in this expression are unique.

Proof This follows from Corollary 3.2.7, where it was shown that the standard q-codeterminants form a free basis for $S_q(n, r)$. ∎

We immediately have the following Corollary:

Corollary 4.2.4

Let $\widehat{Y}^{\lambda}_{i,j}$ be any v-codeterminant.

Then $\widehat{Y}^{\lambda}_{i,j}$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of standard v-codeterminants. The coefficients arising in this expression are unique.

Proof This follows because the element $\widehat{Y}^{\lambda}_{i,j}$ is a power of v times $Y^{\lambda}_{i,j}$, and $q = v^2$. ∎

Definitions

Define the elements $C^{\lambda,i,j}_{\mu,a,b} \in \mathcal{A}$ to be such that

$$\widehat{Y}^{\lambda}_{i,j} = \sum_{\mu,a,b} C^{\lambda,i,j}_{\mu,a,b} \widehat{Y}^{\mu}_{a,b},$$

where the sum is taken over all triples (μ, a, b) such that $Y^{\mu}_{a,b}$ is a standard codeterminant.

Let $Y^{\lambda}_{i,j}$ be a codeterminant, equal to $e_A e_{A'}$. If A is lower triangular and A' is upper triangular, then we say $Y^{\lambda}_{i,j}$ and $\widehat{Y}^{\lambda}_{i,j}$ are *distinguished.*

Note that any standard codeterminant is distinguished. This follows from the correspondence between the $\xi_{i,j}$ and the e_A, and the fact that in a standard tableau, all entries in the i-th row are greater than or equal to i.

Theorem 4.2.5 Any element of the basis B of U which does not map to zero maps to a distinguished v-codeterminant. Every distinguished v-codeterminant is the image of exactly one element of B in a natural way.

Proof From our earlier analysis of $\ker\theta \cap U$, we find that the only elements of B which do not map to zero under θ are of form

$$b = \prod_{\alpha\in\Phi^+} F_\alpha^{(c_\alpha)} \prod_{i=1}^{n} \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix} \prod_{\alpha\in\Phi^+} E_\alpha^{(b_\alpha)},$$

where $\sum_\alpha c_\alpha \leq r$, $\sum_\alpha b_\alpha \leq r$ and $\sum_i t_i = r$. We know that the element $\prod_{i=1}^n \begin{bmatrix} K_i;0 \\ t_i \end{bmatrix}$ maps under θ to $[D]$, where $D = \mathrm{diag}(t_1, \ldots, t_n)$, and that $[D][D] = [D]$. Thus $\theta(b) = \theta(b^-)\theta(b^+)$, where b^- is given by

$$\prod_{\alpha\in\Phi^+} F_\alpha^{(c_\alpha)} \prod_{i=1}^{n} \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix},$$

and b^+ is given by

$$\prod_{i=1}^{n} \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix} \prod_{\alpha\in\Phi^+} E_\alpha^{(b_\alpha)},$$

so in particular $\theta(b^-) \neq 0 \neq \theta(b^+)$. We know from our study of $U^{\leq 0}$ and $U^{\geq 0}$ that if $\theta(b^-) \neq 0$ then $\theta(b^-) = [L]$ for some lower triangular matrix L. Similarly $\theta(b^+) = [U]$ for some upper triangular matrix U. The number t_i is thus the sum of the entries in row i of U, and also the sum of the entries in column i of L. This means that in this case $e_L e_U = \xi_{a,b}\xi_{b,c}$ for suitable $a, b, c \in I(n, r)$, and we see that this quantity is nonzero because Schur's product rule shows that it is nonzero in the classical case. Therefore $\theta(b) = [L][U]$ is a distinguished v-codeterminant.

Conversely, any distinguished v-codeterminant $[L][U]$ such that $[L][U] \neq 0$ has a unique element of B which maps to it, by reversing the previous argument. (The integers in the expression of b determine the coefficients of the matrices L and U, and vice versa.) ∎

Definition For each distinguished v-codeterminant $\widehat{Y}^\lambda_{i,j} = [L][U]$, we define the element $\Upsilon^\lambda_{i,j}$ to be that element of B (as above) which maps under θ to $[L][U]$.

Corollary 4.2.6 (kernel of θ) The kernel of $\theta : U(gl_n) \to S_v(n, r)$ has as a basis all elements of B which map to zero under θ together with all elements of form

$$\Upsilon^\lambda_{i,j} - \sum_{\mu,a,b} C^{\lambda,i,j}_{\mu,a,b} \Upsilon^\mu_{a,b},$$

where $Y^\lambda_{i,j}$ is a distinguished but non-standard codeterminant and $Y^\mu_{a,b}$ is a standard codeterminant.

Proof This follows from the straightening formula for v-codeterminants applied to the set of distinguished v-codeterminants and the fact that each element of B maps to zero or to a distinguished v-codeterminant.

∎

Corollary 4.2.7 (explicit surjectivity) Let $[A]$ be a basis element of $S_v(n, r)$, where e_A corresponds to $\xi_{i,j}$. (We assume, as we may, that $j_a \leq j_b$ whenever $a < b$.) Then the element of U given by

$$\sum_{\mu,a,b} C^{\mathrm{wt}(j),i,j}_{\mu,a,b} \Upsilon^{\mu}_{a,b},$$

where the sum is taken over all (μ, a, b) such that $Y^{\mu}_{a,b}$ is standard, maps under θ to $[A]$.

Proof The element $[A]$ is equal to the codeterminant $\widehat{Y}^{\mathrm{wt}(j)}_{i,j}$, because $e_A = \xi_{i,j} = \xi_{i,j}\xi_j$ and $j = \ell(\mathrm{wt}(j))$. This is equal, by the straightening formula for v-codeterminants, to

$$\sum_{\mu,a,b} C^{\mathrm{wt}(j),i,j}_{\mu,a,b} \widehat{Y}^{\mu}_{a,b}.$$

The claim now follows from the definition of the elements Υ. ∎

5. q-Weyl modules and q-codeterminants

The finite-dimensional representations of the Schur algebra are largely governed by the behaviour of the so-called *Weyl modules*. It is well known that each Weyl module has a basis which is indexed by the set of standard tableaux of a fixed shape.

Dipper and James [DJ3] showed that there exist q-analogues of Weyl modules (known as q-Weyl modules) whose top quotients form a complete set of irreducible representations of $S_q(n, r)$ over a field. They also described an analogue of the "semistandard basis theorem" (which we shall call the *standard basis theorem*, since involves the standard tableaux) for Weyl modules.

We show in this chapter that the standard basis theorem for q-Weyl modules can be rederived more simply by using the theory of quantized codeterminants. We shall also make use of the description of the q-Schur algebra as a quotient of the quantized enveloping algebra $U(gl_n)$ in §4. This enables us to use the theory of quantum groups to prove the standard basis theorem and provides the link between the representation theory of the quantized enveloping algebra with that of the q-Schur algebra.

As in §4, we work with the integral forms of $U(gl_n)$, U^-, U^0 and U^+.

We see from Corollary 4.2.6 that U has a basis B of the form $B_0 \dot{\cup} B_c$, where B_0 is a basis of $U \cap \ker(\theta)$, and the elements B_c are bijectively mapped by θ to the elements of $S_v(n, r)$ corresponding to standard v-codeterminants. We will use this basis of U in this chapter. (Note that the basis B in §5 differs from the basis $B = B^- \otimes B^0 \otimes B^+$ that was used in §4.)

5.1 Quantized Left Weyl Modules

We now define a certain family of left U-modules, W_v^λ, which are parametrised by the elements $\lambda \in \Lambda^+$. We shall refer to W_v^λ as the *left v-Weyl module of shape* λ. It will turn out that W_v^λ is also a $S_v(n, r)$-module in a natural way, and that our v-Weyl modules are essentially the same as Dipper and James' q-Weyl modules.

Until further notice, we shall assume that $n \geq r$, but this will turn out not to be a serious restriction.

Definitions

Let c be an integer satisfying $0 < c \leq r$. Define the element z_c of $V^{\otimes c}$ to be

$$\sum_{\pi \in \mathcal{S}_c} (-v)^{-\ell(\pi)} e_{\pi(1)} \otimes \cdots \otimes e_{\pi(c)}.$$

We define the left U-module M_c as $U.z_c$.

Lemma 5.1.1 The module M_c is a highest weight U-module with highest weight vector z_c. That is,

(a) $E_i.z_c = 0$ for all i;

(b) $K_i.z_c = v^{m(i)} z_c$ for all i and for some function m (called the *highest weight* of the module) from $\{1, \ldots, n\}$ to $\mathbb{Z}$;

(c) $M_c = U^- z_c$.

Proof We first note from the nature of the comultiplication Δ that if $i \geq c$ then $E_i.e_h = 0$ for $h \leq c$, and hence, by the properties of $\Delta^{(c-1)}$, $E_i.z_c = 0$. We may therefore assume that $i < c$.

We define an equivalence relation, $\sim$, on the tensors b appearing in the definition of z_c, by stipulating that $b \sim b'$ if and only if $b = b'$ or b' can be obtained from b by exchanging the positions of e_i and e_{i+1} as they appear in b. (Notice that they must both appear, since $i < c$.) Note that each $\sim$-equivalence class has exactly two elements. It is now enough to show that E_i acts as zero on elements of $V^{\otimes c}$ of the form

$$(-v)^{-\ell(\pi)} e_{\pi(1)} \otimes \cdots \otimes e_{\pi(c)} + (-v)^{-\ell(\sigma)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(c)},$$

where the tensors occurring with nonzero coefficients are related via $\sim$, and $\ell(\sigma) = \ell(\pi) + 1$. This follows immediately from the definition of the comultiplication. As an illustration of the principle involved, we find the action of E_1 on z_2.

$$\begin{aligned} E_1.z_2 &= E_1.(e_1 \otimes e_2 - v^{-1} e_2 \otimes e_1) \\ &= (1 \otimes E_1).(e_1 \otimes e_2) - (E_1 \otimes K_1 K_2^{-1}).(v^{-1} e_2 \otimes e_1) \\ &= e_1 \otimes e_1 - v.v^{-1} e_1 \otimes e_1 \\ &= 0. \end{aligned}$$

This deals with the proof of (a).

Notice that z_c is a sum of multiples of certain basis elements b of $V^{\otimes c}$. The number of times a particular e_h appears in such a basis element b does not depend on which b we choose; this is immediate from the definition of z_c. By using the comultiplication Δ on K_i and acting it on b, we find that K_i acts on b via

$$K_i.b = v^{m(i)}b,$$

where $m(h)$ is the number of times e_h occurs in the tensor b. This also proves that $K_i.z_c = v^{m(i)}z_c$, which is enough to prove (b).

The proof of (c) is immediate from (a) and (b) and the fact that $U \cong U^- \otimes U^0 \otimes U^+$. ∎

Definitions Let $\lambda \in \Lambda^+$. Define z_λ to be the element of $V^{\otimes r}$ given by

$$z_{c_1} \otimes \cdots \otimes z_{c_s},$$

where $c_h = \lambda'_h$, and s is maximal subject to the condition that $\lambda'_s \neq 0$.

Define W_v^λ, the left v-Weyl module of shape λ, to be $U.z_\lambda$.

Lemma 5.1.2 The module W_v^λ is a highest weight U-module of highest weight λ with highest weight vector z_λ.

Proof This follows from Lemma 5.1.1. We act the E_i on z_λ via $\Delta^{(s-1)}$, and Lemma 5.1.1 shows that this is zero.

To see that $K_i.z_\lambda = v^{\lambda(i)}z_\lambda$, we use the techniques of Lemma 5.1.1 (b) and count the number of e_h appearing in each tensor b of z_λ, for each h.

The fact that $W_v^\lambda = U^-.z_\lambda$ is now clear. ∎

Definition Let B_λ be the subset of B_c consisting of all elements of the form

$$\Upsilon = F_{\gamma_N}^{(c_N)} \cdots F_{\gamma_1}^{(c_1)} \begin{bmatrix} K_1; 0 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ \lambda_n \end{bmatrix}.$$

Here, the λ_i are the parts of the (fixed) element $\lambda \in \Lambda^+$, and the c_i are nonnegative integers. As we have assumed that $\Upsilon \in B_c$, we have certain restrictions on the values of the c_i. Denoting by $c_{j,i}$ the element c_h corresponding to the root vector $F_{\alpha(i,j)}$, it can be shown (from the definition of B_c) that necessary and sufficient conditions are that

$$c_{j,i} + (c_{j+1,i} - c_{j+1,i+1}) + \cdots + (c_{n,i} - c_{n,i+1}) \leq \lambda_i - \lambda_{i+1}$$

whenever $1 \leq i < j \leq n$. Note that a similar condition can be found in [CL, (47)].

Note that if $\Upsilon \in B_\lambda$ then $\theta(\Upsilon) = [A][D]$ where $[A][D]$ is a standard v-codeterminant, and $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. All standard codeterminants of form $[A][D]$, where $[D]$ is as above, appear as the image of a unique element $\Upsilon \in B_\lambda$.

Lemma 5.1.3 All elements of $B \backslash B_\lambda$ annihilate z_λ on the left.

Proof It is clear that all elements of B_0 act as zero on tensor space via Δ, and, in particular, annihilate z_λ.

Next, consider elements Υ of B_c which map under θ to standard codeterminants $[A][A']$ where A' is upper triangular but not diagonal. We find from the proof of Theorem 4.2.5 that Υ is naturally expressible in the form $u.u^+$, where $u \in U$ and u^+ is a monomial (including divided powers) in the elements E_α. We see by an easy induction on $h(\alpha)$ that E_α is naturally expressible as a homogeneous polynomial in the E_i of degree $h(\alpha)$. (For example, $E_{\alpha(1,2)} = E_1E_2 - v^{-1}E_2E_1$, which is a homogeneous polynomial in E_1 and E_2 of degree 2.) Since we know each E_i annihilates z_λ (from Lemma 5.1.2), we find that each E_α also annihilates z_λ. Hence Υ annihilates z_λ.

It follows from Corollary 4.1.10 and Theorem 2.2.7 that an element Υ of the form

$$\begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix}$$

will act as the identity on z_λ if $t_h = \lambda_h$ for all h, and will annihilate z_λ otherwise. (Formulated in the notation of Dipper and James, this means that $\phi^1_{t,t}.z_\lambda = \delta_{t,\lambda}z_\lambda$.)

These observations suffice to complete the proof. ∎

Definition

Let B'_λ be the set of elements

$$\Upsilon' = F^{(c_N)}_{\gamma_N} \cdots F^{(c_1)}_{\gamma_1}$$

such that

$$\Upsilon = F^{(c_N)}_{\gamma_N} \cdots F^{(c_1)}_{\gamma_1} \begin{bmatrix} K_1; 0 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ \lambda_n \end{bmatrix} \in B_\lambda.$$

Proposition 5.1.4 The set $\{b.z_\lambda : b \in B_\lambda\}$ is a free $\mathcal{A}$-basis for W^λ_v.

Proof The fact that the set given is a spanning set follows from the definition $W^\lambda_v = U.z_\lambda$, the fact that B is an $\mathcal{A}$-basis for U and Lemma 5.1.3.

To prove independence, we compare this situation with its classical counterpart, which was studied in [CL].

We know (see e.g. the proof of Lemma 5.1.3 above) that the element

$$\begin{bmatrix} K_1;0 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} K_n;0 \\ \lambda_n \end{bmatrix}$$

of U acts as the identity on z_λ. This implies that the family $\{b.z_\lambda : b \in B'_\lambda\}$ is equal (as a subset of $V^{\otimes r}$) to the family $\{b.z_\lambda : b \in B_\lambda\}$.

By specialising v to 1, and comparing with [CL, p. 218], we find that the set $\{b.z_\lambda : b \in B'_\lambda\}$ is (canonically) equal to Carter and Lusztig's free $\mathbb{Z}$-basis for the classical left Weyl module. (The order Carter and Lusztig impose on their root vectors in U^- happens to agree exactly with ours.)

Now consider the quantized case again. Suppose

$$\left(\sum_{i \in B_\lambda} c_i \Upsilon_i \right) .z_\lambda = 0,$$

where the c_i lie in $\mathcal{A}$. Suppose the c_i are not all 0. We may assume (by multiplying throughout by a suitably large power of v) that the c_i are elements of $\mathbb{Z}[v]$.

If $c_i(1) = 0$ (i.e. if c_i specialises to 0) then $v - 1$ divides c_i over the ring $\mathbb{Q}[v]$. Since $\mathbb{Z}[v]$ is a unique factorisation domain, we can invoke Gauss' Lemma and argue that $v - 1$ divides c_i over the ring $\mathcal{A}$. This implies that we may assume $c_i(1) \neq 0$ for some i, otherwise we could divide the equation throughout by $v - 1$ until this was no longer possible. We now replace v by 1. This contradicts Carter and Lusztig's basis theorem for Weyl modules. We conclude that the set that we give is indeed a free $\mathcal{A}$-basis. ∎

5.2. Quantized Right Weyl Modules

We can also introduce a second type of Weyl module. This is a right U-module which is very closely related to the Weyl modules of §5.1. Since all the proofs are analogous to those in §5.1, we shall only sketch proofs in this section.

Definitions Let the generators of U act on V on the right via

$$\begin{aligned} e_i E_j &= \delta_{i,j} e_{i+1}, \\ e_{i+1} F_j &= \delta_{i,j} e_i, \\ e_i K_j &= v^{\delta_{i,j}} e_i, \\ e_i K_j^{-1} &= v^{-\delta_{i,j}} e_i, \end{aligned}$$

It is easy to check that this definition extends (by multiplication in U and linearity) to give a well-defined action of U on V. This can be checked from the relations in U, or from the fact that

$e.u = e.\theta_1(u)$ where $\theta_1 : U \to S_v(n,1)$. Extend this action to a right action of U on $V^{\otimes r}$ via the comultiplication Δ.

We now define ${}^{\lambda}_{v}W$, the right v-Weyl module of shape λ, to be $z_\lambda.U$.

Lemma 5.2.1 The module ${}^{\lambda}_{v}W$ is a lowest-weight right U-module with lowest weight λ and lowest weight vector z_λ.

Proof We need to prove the following:

(a) $z_\lambda F_h = 0$ for all $1 \leq h < n$.

(b) $z_\lambda K_h = v^{\lambda(h)} z_\lambda$.

(c) ${}^{\lambda}_{v}W = z_\lambda.U^+$.

The proof of (c) follows from the proofs of (a) and (b), as in the proof of Lemma 5.1.2. The proof of (b) is almost identical to the corresponding part of Lemma 5.1.2. It remains to prove (a). As in Lemma 5.1.2, we find it is enough to check that F_h annihilates $z_c \in V^{\otimes c}$ when acting via Δ. The proof now follows by an analogous argument to that of the proof of Lemma 5.1.1, by considering $\sim$-equivalence classes of basis elements. We provide a simple example to illustrate this:

$$\begin{aligned} z_2.F_1 &= (e_1 \otimes e_2 - v^{-1} e_2 \otimes e_1).F_1 \\ &= (e_1 \otimes e_2).(K_1^{-1}K_2 \otimes F_1) - (v^{-1}e_2 \otimes e_1).(F_1 \otimes 1) \\ &= v^{-1}e_1 \otimes e_1 - v^{-1}e_1 \otimes e_1 \\ &= 0. \end{aligned}$$

This is enough to complete the argument. ∎

We now wish to find a "standard basis" of the quantized right Weyl modules, ${}^{\lambda}_{v}W$. We denote by W_λ and ${}_{\lambda}W$ the left (respectively right) classical Weyl modules which are obtained by replacing v by 1.

Lemma 5.2.2 As subsets of $V^{\otimes r}$, W_λ and ${}_{\lambda}W$ are identical.

Proof Notice that the action of U^0 on V in the quantized case on the left is the same as the action on the right. The same holds for the actions of U^0 on $V^{\otimes r}$. This implies that $\mathcal{U}^0$ acts on $V^{\otimes r}$ on the left in exactly the same way it acts on the right.

The action of E_h on $V^{\otimes r}$ on the left, in the classical case, is the same as the action of F_h on $V^{\otimes r}$ on the right. This is true because in the classical case, Δ is cocommutative. Similarly, the action of F_h on the left is the same as the action of E_h on the right.

In a similar manner, one can prove that the action of E_α on the left is the same as the action of F_α on the right, and vice versa. It now follows that $\mathcal{U}.z_\lambda$ and $z_\lambda.\mathcal{U}$ are identical. ∎

Corollary 5.2.3 The module ${}_\lambda W$ is a free $\mathbb{Z}$-module whose rank is equal to the number of standard tableaux of shape λ with entries from 1 to n.

Proof This is immediate from Carter and Lusztig's corresponding classical result for left Weyl modules, and Lemma 5.2.2. ∎

Definition Let ${}_\lambda B$ be the subset of B_c consisting of all elements of the form

$$\Upsilon = \begin{bmatrix} K_1; 0 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ \lambda_n \end{bmatrix} E_{\beta_N}^{(b_N)} \cdots E_{\beta_1}^{(b_1)}.$$

Here, the λ_i are the parts of the (fixed) element $\lambda \in \Lambda^+$, and the b_i are nonnegative integers. As we have assumed that $\Upsilon \in B_c$, we have certain restrictions on the values of the b_i. Denoting by $b_{i,j}$ the element b_h corresponding to the root vector $E_{\alpha(i,j)}$, it can be shown (from the definition of B_c) that necessary and sufficient conditions are that

$$b_{i,j} + (b_{i,j+1} - b_{i+1,j+1}) + \cdots + (b_{i,n} - b_{i+1,n}) \leq \lambda_i - \lambda_{i+1}$$

whenever $1 \leq i < j \leq n$.

Note that if $\Upsilon \in {}_\lambda B$ then $\theta(\Upsilon) = [D][A]$ where $[D][A]$ is a standard v-codeterminant, and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. All standard codeterminants of form $[D][A]$, where $[D]$ is as above, appear as the image of a unique element $\Upsilon \in {}_\lambda B$.

Note

Let θ_1 be the homomorphism from U to the matrix ring M_n which sends E_h to $e_{h,h+1}$, F_h to $e_{h+1,h}$, K_h to $v.e_{h,h}$ and K_h^{-1} to $v^{-1}.e_{h,h}$. Define γ_r to be the map from U to $T^r(M_n)$ given by

$$\gamma_r := \underbrace{(\theta_1 \otimes \cdots \otimes \theta_1)}_{r \text{ times}}.$$

Then we see that our left action of U on $V^{\otimes r}$ can be expressed as $u.e = \gamma_r(u).e$, where the second action is the natural left action of $T^r(M_n)$ on $V^{\otimes r}$. Similarly, our right action of U on $V^{\otimes r}$ can be expressed as $e.u = e.\gamma_r(u)$, where the second action is the natural right action of $T^r(M_n)$ on $V^{\otimes r}$.

Lemma 5.2.4 All elements of $B\backslash_\lambda B$ annihilate z_λ on the right.

Proof Let θ_r be the usual homomorphism from U to $S_v(n,r)$. It is a corollary of the proof of Theorem 2.2.7 that $\ker \gamma_r = \ker(\theta)$, because $\gamma_r(u) = \gamma(\theta(u))$ for the monomorphism γ described in §2.2. It is clear that the left and right actions of $T^r(M_n)$ on $V^{\otimes r}$ are faithful, so it follows that the kernels of the left and right actions of U on $V^{\otimes r}$ are both equal to $\ker(\theta)$. It now follows that all elements of B_0 annihilate z_λ on the right.

Next, consider elements Υ of B_c which map under θ to standard codeterminants $[A][A']$ where A is lower triangular but not diagonal. We find from the proof of Theorem 4.2.5 that Υ is naturally expressible in the form $u^-.u$, where $u \in U$ and u^- is a monomial (including divided powers) in the elements F_α. We see by an easy induction on $h(\alpha)$ that F_α is naturally expressible as a homogeneous polynomial in the F_i of degree $h(\alpha)$. Since we know each F_i annihilates z_λ (from Lemma 5.2.1), we find that each F_α also annihilates z_λ. Hence Υ annihilates z_λ.

It follows from the proof of Lemma 5.1.3 and the similarities between the left and right actions of U^0 on $V^{\otimes r}$ that all elements Υ of the form

$$\begin{bmatrix} K_1;0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n;0 \\ t_n \end{bmatrix}$$

will act as the identity on z_λ if $t_h = \lambda_h$ for all h, and will annihilate z_λ otherwise.

These observations suffice to complete the proof. ∎

We are now ready to prove the standard basis theorem for quantized right Weyl modules.

Proposition 5.2.5 The set $\{z_\lambda.b : b \in {}_\lambda B\}$ is a free $\mathcal{A}$-basis for ${}_v^\lambda W$.

Proof The fact that the set given is a spanning set follows from the definition $W_v^\lambda = z_\lambda.U$, the fact that B is an $\mathcal{A}$-basis for U and Lemma 5.2.4.

We now tackle independence. By using an argument based around Gauss' Lemma (as in Lemma 5.1.4), we find that if the given set is not an $\mathcal{A}$-basis in the quantized case, then the corresponding set in the classical case fails to be a $\mathbb{Z}$-basis. However, the set ${}_\lambda B$ has the same cardinality as the rank of ${}_\lambda W$ as a $\mathbb{Z}$-module (as shown by Corollary 5.2.3). It now follows that the given elements are independent. ∎

Before stating the main result of this chapter, we note that the modules ${}_v^\lambda W$ and W_v^λ are identical as subsets of tensor space. This is not required for the proof of the standard basis theorem, but we provide a proof for completeness and for interest's sake.

Definitions

The *weight* of a basis vector $b = e_{i_1} \otimes \cdots \otimes e_{i_r}$ of $V^{\otimes r}$ is defined to be the n-tuple $\mu := (\mu_1, \dots, \mu_n)$ where μ_h is the multiplicity with which the vector e_h occurs in the basis element. (This implies that $K_h.b = v^{\mu_h}.b$.)

The subspace $V^{\mu,r}$ of $V^{\otimes r}$ is that which is spanned by all basis elements of weight μ. We call such a subspace a *weight space*.

Lemma 5.2.6 If b is an element of $V^{\mu,r}$, then the vectors $E_h.b$, $F_h.b$, $b.E_h$ and $b.F_h$ all lie in weight spaces.

The vectors $F_h.b$ and $b.E_h$ lie in the same weight space, and the vectors $E_h.b$ and $b.F_h$ lie in the same weight space.

Proof The first assertion is well-known and is immediate from the definitions of the actions of E_h and F_h on the left and on the right of $V^{\otimes r}$. For example, if $1 \leq h < n$, $b \in V^{\mu,r}$ and ν is the weight given by

$$\nu_i = \begin{cases} \mu_i + 1 & \text{if } i = h, \\ \mu_i - 1 & \text{if } i = h+1, \\ \mu_i & \text{otherwise,} \end{cases}$$

then we find that $E_h.b$ lies in $V^{\nu,r}$. The other three cases follow a similar format.

The proof of the second assertion also follows quickly. ∎

Definitions

The *μ-weight space* of W_v^λ (respectively, ${}_v^\lambda W$) is given by $W_v^\lambda \cap V^{\mu,r}$ (respectively, ${}_v^\lambda W \cap V^{\mu,r}$).

If h is an integer satisfying $1 \leq h < n$, then the function $f_h : \Lambda(n,r) \to \mathbb{Z}$ is defined as $f_h(\mu) := \mu_h - \mu_{h+1}$.

We write $[e_{i_1} \wedge \cdots \wedge e_{i_c}]$ as shorthand for the *quantized exterior power* given by

$$\sum_{\sigma \in \mathcal{S}_c} (-v)^{-\ell(\sigma)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_c)},$$

where the symmetric group is acting by the usual place permutation.

Let T be a tableau with r boxes, and write $T(i,j)$ for the j-th entry in the i-th row of T. Denote the length of the column h of T by $C(h)$, and denote the number of columns of T by c. We define the element $L(T)$ of $V^{\otimes r}$ by

$$L(T) := [e_{T(1,1)} \wedge \cdots \wedge e_{T(C(1),1)}] \otimes [e_{T(1,2)} \wedge \cdots \wedge e_{T(C(2),2)}] \otimes \cdots \otimes [e_{T(1,c)} \wedge \cdots \wedge e_{T(C(c),c)}].$$

Define $L_{n,r}$ to be the subspace of $V^{\otimes r}$ (over the field $\mathbb{Q}(v)$) spanned by all $L(T)$ as T ranges over all tableaux with r boxes, entries from 1 to n, and strictly increasing columns. If $\mu \in \Lambda(n,r)$, we define $L^{\mu}_{n,r}$, the μ-weight space of $L_{n,r}$, to be $L_{n,r} \cap V^{\mu,r}$. Notice that any $L(T)$ lies in some weight space of $L_{n,r}$.

Define the function $\lambda_a(T, b)$ (where $a, b \in \mathbb{N}$ and T is a tableau) to be the number of occurrences of the number a in columns strictly to the left of column b.

Define the function $\rho_a(T, b)$ (where $a, b \in \mathbb{N}$ and T is a tableau) to be the number of occurrences of the number a in columns strictly to the right of column b.

Define $\kappa_h \subset \mathbb{N}$ to be the set of numbers c such that column c of T contains an entry h.

Let $a, b, c \in \mathbb{N}$ and let T be a tableau. The tableau $T' = T(c, a, b)$ is defined to be that tableau of the same shape as T with the topmost occurrence of the number a in column c replaced by an occurrence of b.

Example

Let $r = 7, n = 4$ and T be the tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} .$$

Then $L(T)$ is given by

$$[e_1 \wedge e_3 \wedge e_4] \otimes [e_2 \wedge e_3] \otimes e_3 \otimes e_4,$$

and $[e_1 \wedge e_3 \wedge e_4]$ expands to

$$e_1 \otimes e_3 \otimes e_4 - v^{-1} e_1 \otimes e_4 \otimes e_3 - v^{-1} e_3 \otimes e_1 \otimes e_4 + v^{-2} e_3 \otimes e_4 \otimes e_1 + v^{-2} e_4 \otimes e_1 \otimes e_3 - v^{-3} e_4 \otimes e_3 \otimes e_1.$$

The value of $\lambda_3(T, 3)$ is 2, and the value of $\rho_4(T, 2)$ is 1.

The tableau $T' = T(2, 3, 4)$ is given by

$$T' = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} .$$

We now present multiplication formulae for the left and right actions of the E_h and F_h on an element $L(T)$.

Lemma 5.2.7 Consider $L_{n,r}$ as a subspace of $V^{\otimes r}$ under the usual action of U, and let $L(T) \in L_{n,r}$ for some suitable tableau T. Define μ to be the weight such that $L(T)$ is in the μ-weight space of $L_{n,r}$. Then the following identities hold:

$$F_h.L(T) = \sum_{c \in \kappa_h \backslash \kappa_{h+1}} v^{-\lambda_h(T,c)+\lambda_{h+1}(T,c)} L(T(c,h,h+1)), \tag{1}$$

$$E_h.L(T) = \sum_{c \in \kappa_{h+1} \backslash \kappa_h} v^{\rho_h(T,c)-\rho_{h+1}(T,c)} L(T(c,h+1,h)), \tag{2}$$

$$L(T).F_h = \sum_{c \in \kappa_{h+1} \backslash \kappa_h} v^{-\lambda_h(T,c)+\lambda_{h+1}(T,c)} L(T(c,h+1,h)), \tag{3}$$

$$L(T).E_h = \sum_{c \in \kappa_h \backslash \kappa_{h+1}} v^{\rho_h(T,c)-\rho_{h+1}(T,c)} L(T(c,h,h+1)). \tag{4}$$

Therefore, the following identities also hold:

$$v^{f_h(\mu)-1} F_h.L(T) = L(T).E_h, \tag{5}$$

$$v^{f_h(\mu)+1} L(T).F_h = E_h.L(T). \tag{6}$$

Proof The proof of (5) follows from (1) and (4) and the fact that for any $c \in \kappa_h \backslash \kappa_{h+1}$ we have

$$f_h(\mu) = \lambda_h(T,c) + 1 + \rho_h(T,c) - \lambda_{h+1}(T,c) - \rho_{h+1}(T,c).$$

The proof of (6) follows from (2) and (3) and the fact that for any $c \in \kappa_{h+1} \backslash \kappa_h$ we have

$$f_h(\mu) = \lambda_h(T,c) + \rho_h(T,c) - \lambda_{h+1}(T,c) - 1 - \rho_{h+1}(T,c).$$

We must now establish the truth of (1), (2), (3) and (4). The actions of E_h and F_h on $L(T)$ (on the left or on the right) can in some sense be regarded as actions on the columns of T and extended via the comultiplication Δ. The functions λ and ρ are exactly what is needed to make the lemma work in (1), (2), (3) and (4), provided that the lemma remains true for tableaux which consist of a single column, so we concentrate on this case.

We are now assuming that T has one column. Since we assume $L(T)$ is defined, the entries of T strictly increase down the columns. If T contains the number $h+1$ but not h, or if it does not contain $h+1$ at all, (2) and (3) are easily verified. Similarly if T contains the number h but not $h+1$, or if it does not contain h at all, (1) and (4) are easily verified.

The only difficult cases are when T contains both h and $h+1$. We wish to show that E_h and F_h act as zero on the left and on the right. The tableau

$$T = \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array},$$

where $a = h$ and $b = h + 1$, illustrates all the important features of the general case. There are four cases to check, which we present below. We assume $h = 1$ in the examples below for clarity, although this is not important.

$$\begin{aligned}
E_1.(e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1) &= (1 \otimes E_1 + E_1 \otimes K_1K_2^{-1}).(e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1) \\
&= e_1 \otimes e_1 - v.v^{-1}e_1 \otimes e_1 \\
&= 0 \\
F_1.(e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1) &= (F_1 \otimes 1 + K_1^{-1}K_2 \otimes F_1).(e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1) \\
&= e_2 \otimes e_2 - v.v^{-1}e_2 \otimes e_2 \\
&= 0 \\
(e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1).E_1 &= (e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1).(1 \otimes E_1 + E_1 \otimes K_1K_2^{-1}) \\
&= e_1 \otimes e_1 - v.v^{-1}e_1 \otimes e_1 \\
&= 0 \\
(e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1).F_1 &= (e_1 \otimes e_2 - v^{-1}e_2 \otimes e_1).(F_1 \otimes 1 + K_1^{-1}K_2 \otimes F_1) \\
&= e_2 \otimes e_2 - v.v^{-1}e_2 \otimes e_2 \\
&= 0
\end{aligned}$$

These observations suffice to complete the proof. ∎

Proposition 5.2.8 As sets, $W_v^\lambda = {}_v^\lambda W$.

Proof Denoting the basic λ-tableau by T, we have $z_\lambda = L(T)$. It is clear from Lemma 5.1.1 (c), Lemma 5.2.1 (c) and the definitions of left and right v-Weyl modules that $W_v^\lambda = U^-.L(T)$ and ${}_v^\lambda W = L(T).U^+$.

Since U^- has a spanning set consisting of monomials (with divided powers) in the F_i, viz. $F_{i_s}^{(c_s)} \cdots F_{i_1}^{(c_1)}$, we find that a spanning set for W_v^λ is given by

$$F_{i_s}^{(c_s)} \cdots F_{i_1}^{(c_1)}.L(T)$$

Similarly, ${}_v^\lambda W$ has a spanning set consisting of elements given by

$$L(T).E_{i_1}^{(c_1)} \cdots E_{i_s}^{(c_s)}.$$

To prove equality of the two modules, it is enough to show that

$$F_{i_s}^{(c_s)} \cdots F_{i_1}^{(c_1)}.L(T) = v^* L(T).E_{i_1}^{(c_1)} \cdots E_{i_s}^{(c_s)}$$

for some suitable integer represented by $*$. Since the divided powers are the same on both sides of the equation, this reduces to proving that

$$F_{i_s}^{c_s} \cdots F_{i_1}^{c_1}.L(T) = v^* L(T).E_{i_1}^{c_1} \cdots E_{i_s}^{c_s}.$$

Since $L(T)$ lies in a weight space of $L_{n,r}$ (and hence of each v-Weyl module), the proof of the proposition now follows from repeated application of Lemma 5.2.6 and Lemma 5.2.7, equation (5).

∎

5.3. q-Codeterminants and q-Weyl modules

We are now ready to explain the relationship between q-codeterminants and q-Weyl modules.

Definitions

For each $\lambda \in \Lambda^+$, define the left q-Weyl module of shape λ, W_q^λ, to be $S_q(n,r).z_\lambda$.

For each $\lambda \in \Lambda^+$, define the right q-Weyl module of shape λ, ${}^\lambda_q W$ to be $z_\lambda.S_q(n,r)$.

Note that the left and right v-Weyl modules can be recovered by appropriate tensoring over $\mathcal{A}$.

Let C^λ be the set of standard q-codeterminants $e_A e_D$ such that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Let ${}^\lambda C$ be the set of standard q-codeterminants $e_D e_{A'}$ such that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Theorem 5.3.1 (the Standard Basis Theorem for q-Weyl Modules)

(a) The set $\{C^\lambda.z_\lambda : \lambda \in \Lambda^+(n,r)\}$ is a free $\mathbb{Z}[q,q^{-1}]$-basis for W_q^λ.

(b) The set $\{z_\lambda.{}^\lambda C : \lambda \in \Lambda^+(n,r)\}$ is a free $\mathbb{Z}[q,q^{-1}]$-basis for ${}^\lambda_q W$.

Proof Part (a) is a restatement of Proposition 5.1.4, and part (b) is a restatement of Proposition 5.2.5. To see this, we use Theorem 4.2.5 to explain the relationship between PBW-type elements in U with standard codeterminants in $S_q(n,r)$, and also the fact that

$$S_v(n,r) \cong S_q(n,r) \oplus v.S_q(n,r)$$

as a $\mathbb{Z}[q,q^{-1}]$ module. ∎

We need to check that our quantized left Weyl modules agree with those of Dipper and James in [DJ3]. When $\lambda \in \Lambda^+$ and $n > r$, Dipper and James define the (left) q-Weyl module to be the left ideal of $S_q(n,r)$ generated by an element which they call z_λ. This is not obviously the same as our z_λ, although it will turn out that the two are multiples of each other. The Dipper–James definition of z_λ is

$$\phi^1_{\lambda\omega} T_{w_\lambda} y_{\lambda'},$$

where

$$y_{\lambda'} := \sum_{\pi \in \mathcal{S}_{\lambda'}} (-q)^{-\ell(\pi)} T_\pi .$$

The element T_π represents the element $\phi^\pi_{\omega\omega}$ or $\xi_{\omega.\pi,\omega}$ of $S_q(n,r)$, and can be considered to be a basis element of the Hecke algebra. Thus, in the combinatorial set-up of the q-Schur algebra, the element z_λ of Dipper and James is denoted by

$$\xi_{\ell(\lambda),\omega} \xi_{\omega.w_\lambda,w} \sum_{\pi \in \mathcal{S}_{\lambda'}} (-q)^{-\ell(\pi)} \xi_{\omega.\pi,\omega}.$$

The next few results establish the required equivalence.

Lemma 5.3.2 Let $i \in I(n,r)$ (where $n > r$ as usual) and suppose $i_p < i_{p+1}$ for some p. Let s be the simple transposition $(p, p+1)$. Then the identity

$$\xi_{i,\omega} \xi_{\omega.s,\omega} = \xi_{i.s,\omega}$$

holds in $S_q(n,r)$.

Proof

Let e_A, e_P and $e_{A'}$ be the basis elements expressed above as $\xi_{i,\omega}$, $\xi_{\omega.s,\omega}$ and $\xi_{i.s,\omega}$ respectively. We first consider the product $[A][P]$ by using the techniques introduced in §2.2. Using the notation in that chapter, we find that the only relaxed basis element of $T^r(M_n)$ occurring in the product with nonzero coefficient is that corresponding to the basis element $[A']$. It arises from the product

$$v^{-m_A} t_A . T_s \times v^{-m_P} t_P .$$

Note that the condition $i_p < i_{p+1}$ ensures that $s \in \mathcal{D}_A$.

The nature of ω ensures that the integers m_A, m_P and $m_{A'}$ are all equal to zero. The hypothesis on i ensures that the p-th and $(p+1)$-th positions in the tensor t_A are of forms $e_{a,p}$ and $e_{b,p+1}$ respectively, where $a < b$. Since we have

$$e_{a,p} \otimes e_{b,p+1} . T_s = e_{b,p+1} \otimes e_{a,p}$$

by definition of the action of $\mathcal{H}$ on $T^r(M_n)$, we see that $t_{A'}$ occurs with coefficient 1 in the above product.

We now shift our attention to the product $e_A e_P$. We know that $e_B = v^{s_B}[B]$, where

$$s_B := \sum_{1 \leq i,j,k,l \leq n} B_{i,j} B_{k,l},$$

where the sum is taken with the restrictions that restrictions that $i \geq k$ and $j < l$. From this, we see that $s_P = 1$ and $s_{A'} = s_A + 1$. This establishes the desired result that $e_A.e_P = e_{A'}$. ∎

Remarks on Lemma 5.3.2 Notice that the corresponding classical identity is immediate from Schur's product rule. Also notice that the identity in the Hecke algebra $T_{ws} = T_w T_s$ (where $\ell(ws) > \ell(w)$) can be considered to be a special case of this result.

We now describe a submodule of $S_v(n,r)$ which is isomorphic as a left U-module to $T^r(V)$.

Define the linear map $\tau : T^r(V) \to S_v(n,r)$ by

$$\tau(e_{i_1} \otimes \cdots \otimes e_{i_r}) := [A_i],$$

where the (a,b)-entry of A_i is equal to 1 if $a = i_b$ and 0 otherwise.

Lemma 5.3.3 The map τ is a canonical monomorphism of left U-modules.

Proof Note that as usual, $S_v(n,r)$ is made into a U-module via

$$u.s := \theta(u) \times s.$$

Injectivity is clear since the elements $[A]$ form a basis of $S_v(n,r)$. The fact that τ is a module homomorphism follows from the multiplication formulae in [BLM, Lemma 3.4]. (In fact, this is the reason we use the comultiplication Δ and not one of the other possible comultiplications.) ∎

We now exhibit the link between tensor space and the q-tensor space of Dipper and James. In [DJ3], q-tensor space is defined to be spanned by elements of the form $\phi^d_{\mu,\omega}$. In the combinatorial set-up of $S_q(n,r)$, this space is spanned by elements of the form $\xi_{i,\omega}$.

Lemma 5.3.4 Let $i \in I(n,r)$. Write e_i for the tensor $e_{i_1} \otimes \cdots \otimes e_{i_r}$. Then the map τ satisfies

$$\tau(e_i) = v^{-s(i)} \xi_{i,\omega},$$

where $s(i) := |\{a < b : i_a \geq i_b\}|$.

Proof It is easily checked that $e_{A_i} = \xi_{i,\omega}$ under the usual identification. Since $e_A = v^{s_A}[A]$, it is enough, by Lemma 5.3.3, to check that $s_{A_i} = s(i)$, but this is immediate. ∎

Lemma 5.3.5 We have

$$\phi^1_{\lambda\omega} T_{w_\lambda} y_{\lambda'} = \sum_{\pi \in \mathcal{S}_{\lambda'}} (-q)^{-\ell(\pi)} \xi_{\ell(\lambda).w_\lambda.\pi,\omega}.$$

Proof It is shown in [DJ3, §3] that w_λ is a double $\mathcal{S}_\lambda$–$\mathcal{S}_{\lambda'}$ coset representative with the property that

$$w_\lambda^{-1}\mathcal{S}_\lambda w_\lambda \cap \mathcal{S}_{\lambda'} = 1.$$

This implies that if $a \in \mathcal{S}_\lambda$ and $b \in \mathcal{S}_{\lambda'}$ then $\ell(aw_\lambda b) = \ell(a) + \ell(w_\lambda) + \ell(b)$. Choose $\pi \in \mathcal{S}_{\lambda'}$. Then $\ell(w_\lambda \pi) = \ell(w_\lambda) + \ell(\pi)$ and $w_\lambda.\pi$ is a distinguished right $\mathcal{S}_\lambda$-coset representative.

We now express $w_\lambda.\pi$ as a product $s_{i_1}\cdots s_{i_m}$ of simple reflections, and use induction on p where $1 \leq p \leq m$ to show that

$$\xi_{\ell(\lambda),\omega}.\xi_{\omega.w_\lambda,w}.\xi_{\omega.\pi,\omega} = \xi_{\ell(\lambda).w_\lambda.\pi,\omega}.$$

(The element $\phi^1_{\lambda,w}$ is equal to $\xi_{\ell(\lambda),\omega}$ under the identification of the two bases.)

Note that since $w_\lambda.\pi$ is a distinguished right $\mathcal{S}_\lambda$-coset representative, it is immediate that $s_{i_1}\cdots s_{i_p}$ is also a distinguished right $\mathcal{S}_\lambda$-coset representative, where $1 \leq p \leq m$.

The case $p = 1$ of the induction is to prove that

$$\xi_{\ell(\lambda),\omega} \times \xi_{\omega.s_{i_1},\omega} = \xi_{\ell(\lambda).s_{i_1},\omega}.$$

This is proved by application of Lemma 5.3.2, which is applicable because s_{i_1} is a distinguished right $\mathcal{S}_\lambda$-coset representative, so if we express s_{i_1} as a transposition (i_1, i_1+1), we find that $\ell(\lambda)_{i_1} < \ell(\lambda)_{i_1+1}$.

The inductive hypothesis is that

$$\xi_{\ell(\lambda).s_{i_1}\cdots s_{i_{p-1}},\omega} \times \xi_{\omega.s_{i_p},\omega} = \xi_{\ell(\lambda).s_{i_1}\cdots s_{i_p},\omega}.$$

Since $s_{i_1}\cdots s_{i_p}$ is a distinguished right $\mathcal{S}_\lambda$-coset representative, we have

$$(\ell(\lambda).s_{i_1}\cdots s_{i_{p-1}})_{i_p} < (\ell(\lambda).s_{i_1}\cdots s_{i_{p-1}})_{i_p+1}.$$

Lemma 5.3.2 is now applicable, and this completes the induction.

The main proof now follows from the discussion preceding Lemma 5.3.2. ∎

Proposition 5.3.6 The module W_q^λ is canonically isomorphic to Dipper and James's module W_λ.

Proof It is enough to show that, using the map τ of Lemma 5.3.4 and the identity in Lemma 5.3.5 to make the relevant identifications, our element z_λ is equal to a power of v times the element $z_\lambda \in S_q(n,r)$ of Dipper and James.

Define k to be the number of pairs $a < b$ such that $\ell(\lambda)_a \geq \ell(\lambda)_b$. Notice that if i is the r-tuple $\ell(\lambda).w_\lambda.\pi$ then the number of pairs $a < b$ such that $i_a \geq i_b$ is equal to $\ell(w_\lambda) + \ell(\pi) + k$, by standard properties of symmetric group actions. We now see, using the function s in Lemma 5.3.4, that

$$s(\ell(\lambda).w_\lambda.\pi) = \ell(w_\lambda) + \ell(\pi) + k.$$

We now see from Lemmas 5.3.4 and 5.3.5 that W_λ is isomorphic as a left U-module to the submodule of tensor space generated by the vector

$$\sum_{\pi \in \mathcal{S}_{\lambda'}} (-q)^{-\ell(\pi)} v^{k+\ell(w_\lambda)+\ell(\pi)} e_{\ell(\lambda).w_\lambda.\pi},$$

Using the fact that $q = v^2$, we find that this is equal to

$$v^{k'} . \sum_{\pi \in \mathcal{S}_{\lambda'}} (-v)^{-\ell(\pi)} e_{\ell(\lambda).w_\lambda.\pi},$$

where $k' = \ell(\omega_\lambda) + k$.

This now shows that Dipper and James's z_λ can be canonically identified, by multiplying by $v^{-k'}$, with our z_λ. This means that the modules W_q^λ and W_λ are isomorphic as U-modules, and hence as $S_q(n,r)$-modules, as required. ∎

5.4. Remarks on the case $n < r$

In the case $n \geq r$, it is shown in [DJ3, Theorem 8.8] that each of the q-Weyl modules W_λ has a unique maximal submodule, which we denote by M_λ. The set of all $F_\lambda := W_\lambda / M_\lambda$ as λ varies over Λ^+ is shown to be a complete and irredundantly described set of absolutely irreducible representations for $S_q(n,r)$ over a field.

Dipper and James also remark that if $n < r$, the complete set of irreducibles is given by

$$\{e.F_\lambda : \lambda \in \Lambda^+(n,r)\},$$

where now $\Lambda^+(n,r)$ is the set of partitions of r into not more than n pieces, F_λ is the top quotient of a q-Weyl module $S_q(N,r)$ for some $N \geq r$, and e is the sum of all idempotent basis elements e_D of $S_q(N,r)$ where D is a diagonal matrix satisfying $D_{i,i} = 0$ for $i > n$.

This construction relies on some canonical embeddings of the q-Schur algebra. The algebra $S_q(n,r)$ embeds in $S_q(N,r)$ via the map $e_A \mapsto e_{\epsilon(A)}$, where the (i,j)-entry of $\epsilon(A)$ is given by $A_{i,j}$ if $i \leq n$ and $j \leq n$, and is zero otherwise. It is trivial to check that this is a monomorphism of algebras. Another way of looking at this is to notice that $S_q(n,r) \cong eS_q(N,r)e$. This means that the q-Weyl modules for $S_q(n,r)$ come from certain q-Weyl modules for $S_q(N,r)$ by restriction.

In a somewhat similar way, the quantized enveloping algebra $U(gl_n)$ embeds in $U(gl_N)$ by the obvious identification of the generators. From the viewpoint of quantized enveloping algebras, the left quantized Weyl modules W_v^λ for $U(gl_n)$ are restrictions of certain left quantized Weyl modules W_v^λ for $U(gl_N)$. The highest weight vectors are the same in both cases. (We have implicitly embedded $\lambda \in \Lambda^+(n,r)$ in $\Lambda^+(N,r)$ by sending λ to $(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$.)

Similar remarks hold for right q-Weyl modules. In this case we find that the complete set of irreducibles for $S_q(n,r)$ is

$$\{{}_\lambda F.e : \lambda \in \Lambda^+(n,r)\},$$

and that the modules ${}_v^\lambda W$ for $U(gl_n)$ arise from the corresponding ones for $U(gl_N)$ by restriction. In the latter case, the lowest weight vectors are identical.

The various versions of the standard basis theorem now carry over essentially unchanged.

6. Cellular inverse limits of q-Schur algebras

It is known that the classical codeterminants of [G2] cast light on the structure of the Schur algebra as a quasi-hereditary algebra (in the sense of Cline, Parshall and Scott).

In this chapter, we make a more detailed study of quantized codeterminants. In §6.1, we strengthen the straightening result given in §3 by proving that any quantized codeterminant of dominant shape λ can be expressed as a linear combination of quantized codeterminants whose shapes dominate λ, in a sense which we will explain. (An analogous result is known (see [M, Proposition 2.6.9]) in the classical case.) This allows one to obtain results concerning the structure constants of the q-Schur algebra with respect to the basis of standard q-codeterminants.

We also show in §6.2 that quantized codeterminants can be used to illustrate the quasi-hereditary structure of the q-Schur algebra and the structure of the q-Schur algebra as a cellular algebra in the sense of Graham and Lehrer [GL]. We study the properties of ideals spanned by codeterminants with shapes lying in a given ascending saturation, which is an ideal of the poset of weights of the q-Schur algebra under the dominance order. In §6.3, we find that, by quotienting out certain of these ideals, one can show that the v-Schur algebra $S_v(n, r+kn)$ has as a quotient an algebra canonically isomorphic to $S_v(n,r)$ for any natural number k; we construct the epimorphism explicitly.

In §6.4, we construct, using the epimorphisms in §6.3, an inverse system of v-Schur algebras $S_v(n,r)$ with fixed values of n. This turns out to be related in an interesting way to the quantized enveloping algebra $U(sl_n)$ and to Lusztig's algebra $\dot{U}$ of type A.

As in §4 and §5, we use the integral forms of quantized enveloping algebras, except where otherwise stated.

6.1. The strong straightening result

In §3, it was proved that the set of standard v-codeterminants (respectively q-codeterminants) formed a free $\mathbb{Z}[v,v^{-1}]$ basis (respectively $\mathbb{Z}[q,q^{-1}]$ basis) of $S_v(n,r)$ (respectively $S_q(n,r)$). We shall prove in this section that the standard codeterminants occurring with nonzero coefficients in the expression for a codeterminant of dominant shape λ have shapes equal to or dominating λ. (This result is well-known in the classical case in the context of bideterminants—see for example [M, Proposition 2.6.9].) It also turns out, although we shall not prove this, that this result remains true for codeterminants of arbitrary shape and a suitable natural extension of the order $\trianglerighteq$ to all of $\Lambda(n,r)$. We shall refer to this result as the *strong straightening result.*

In the proof of this result, we shall require the following basic fact about tableaux.

Lemma 6.1.1 Let T be a standard tableau of shape λ and let $b \in I(n,r)$ be such that $T = T_b^\lambda$. Let $\mu = \mathrm{wt}(b)$. Then $\lambda \trianglerighteq \mu$.

Proof Since T is standard, all the occurrences of i (where $1 \leq i \leq n$) must occur in the first i rows. The result is now immediate from the definition of the dominance order. ∎

We now obtain a partial result for the strong straightening result.

Lemma 6.1.2

a) Let $Y^\lambda_{i,\ell(\lambda)}$ be a q-codeterminant of dominant shape (not necessarily standard). Then any standard q-codeterminant $Y^\nu_{a,b}$ appearing in the expansion of $Y^\lambda_{i,\ell(\lambda)}$ in terms of standard q-codeterminants satisfies $a \sim i$, $b \sim \ell(\lambda)$ and $\nu \trianglerighteq \lambda$. If $\nu = \lambda$ then $b = \ell(\lambda)$.

b) Let $Y^\lambda_{\ell(\lambda),j}$ be a q-codeterminant of dominant shape (not necessarily standard). Then any standard q-codeterminant $Y^\nu_{a,b}$ appearing in the expansion of $Y^\lambda_{\ell(\lambda),j}$ in terms of standard q-codeterminants satisfies $a \sim \ell(\lambda)$, $b \sim j$ and $\nu \trianglerighteq \lambda$. If $\nu = \lambda$ then $a = \ell(\lambda)$.

Proof We only prove a), because the proof of b) follows by symmetry (i.e. the anti-automorphism of $S_q(n,r)$ taking $Y^\mu_{c,d}$ to $Y^\mu_{d,c}$).

The assertions $a \sim i$ and $b \sim \ell(\lambda)$ are immediate from weight space considerations. Consider a typical standard q-codeterminant occurring in this expression and denote it by $Y^\nu_{a,b}$. It follows from Lemma 6.1.1 that $\nu \trianglerighteq \lambda$, because b is of weight λ and $Y^\nu_{a,b}$ is standard.

The proof of the last part follows from the observation that $\ell(\nu)_i \leq b_i$ for all i, because $Y^\nu_{a,b}$ is standard. If we assume $\nu = \lambda$ then $\ell(\nu) \sim b$, which now forces $b_i = \ell(\nu)_i$ for all i, as required. ∎

The above lemma does the basic work for proving the strong straightening result, which we now prove in full.

Proposition 6.1.3 (the Strong Straightening Result)

Let $Y^\lambda_{i,j}$ be a q-codeterminant (not necessarily standard) with the property that $\lambda \in \Lambda^+$. All standard q-codeterminants appearing in the expansion of $Y^\lambda_{i,j}$ in terms of the q-codeterminant basis are of form $Y^\nu_{a,b}$, where $a \sim i$, $b \sim j$ and $\nu \trianglerighteq \lambda$.

Proof It is clear from weight space considerations that $a \sim i$ and $b \sim j$.

To prove the rest of the proposition, we proceed by induction on the partial order $\trianglerighteq$ on the set $\Lambda^+(n,r)$. The base case concerns the weight $\lambda = (r, 0, \ldots, 0)$, which can easily be checked to have the property that $\lambda \trianglerighteq \mu$ for any $\mu \in \Lambda^+$. This case is trivial because any q-codeterminant of shape

λ is automatically standard—this follows from the definitions of standard codeterminants.

We now assume the theorem is true for all q-codeterminants of shape ν, where $\nu \rhd \lambda$. We rewrite $Y^\lambda_{i,j}$ as $Y_{i,\ell(\lambda)}.Y_{\ell(\lambda),j}$ and apply both parts a) and b) of Lemma 6.1.2 to expand each half of the product in terms of standard q-codeterminants. We now have an expression for $Y^\lambda_{i,j}$ in terms of linear combinations of certain products of pairs of standard q-codeterminants. These are of four types, as follows.

1. Products of a term $Y^\lambda_{i',\ell(\lambda)}$ and a term $Y^\nu_{a,b}$, where $\nu \rhd \lambda$.
2. Products of a term $Y^\nu_{a,b}$ and a term $Y^\lambda_{\ell(\lambda),j'}$, where $\nu \rhd \lambda$.
3. Products of a term $Y^\lambda_{i',\ell(\lambda)}$ and a term $Y^\lambda_{\ell(\lambda),j'}$.
4. Products of a term $Y^\nu_{a,b}$ and a term $Y^{\nu'}_{c,d}$, where $\nu \rhd \lambda$ and $\nu' \rhd \lambda$.

Products of type 1 and 2 can be naturally re-expressed as linear combinations of quantized codeterminants of shape ν, where $\nu \rhd \lambda$ as described. For example, in type 1 we re-express

$$Y^\lambda_{i',\ell(\lambda)}.Y^\nu_{a,b} = (\xi_{i',\ell(\lambda)}.\xi_{a,\ell(\nu)}).\xi_{\ell(\nu),b},$$

and expand the parentheses using the product rule for q-Schur algebras. Using the inductive hypothesis, we see that these terms are expressible as sums of *standard* q-codeterminants of shape $\mu \unrhd \nu \rhd \lambda$.

The case of products of type 3 is very simple, because

$$Y^\lambda_{i',\ell(\lambda)}.Y^\lambda_{\ell(\lambda),j'} = Y^\lambda_{i',j'},$$

which is a standard q-codeterminant.

Products of type 4 can be dealt with in the same way as products of type 1 or 2.

This completes the induction and the proof of the proposition. ∎

Remarks

1. The straightening formula given in Theorem 3.2.6 proves that any q-codeterminant of dominant shape λ can be expressed as a sum of standard q-codeterminants of shape μ, where $\mu \succeq \lambda$ for a certain total order $\succ$ refining $\rhd$. The strong straightening result is not immediately clear from using that technique.
2. A similar result can be formulated for q-codeterminants of arbitrary shape, but we shall not need this for our purposes.

Using the strong straightening result and the properties of increasing saturations, we are now able to define a class of ideals of the q-Schur algebra, as follows.

Proposition 6.1.4 Let π be an increasing saturation. Then the ideal $Y(\pi)$ of $S_q(n,r)$ spanned by all codeterminants of shape $\lambda \in \pi$ has as a basis the set of all standard q-codeterminants of shapes μ where $\mu \in \pi$.

Proof It is clear that the space spanned by all the q-codeterminants of shape λ, where $\lambda \in \pi$, forms a two-sided ideal of $S_q(n,r)$, by using the techniques needed to deal with the proofs of cases 1 and 2 in the proof of Proposition 6.1.3. It is immediate from the basis theorem for q-codeterminants that the standard q-codeterminants with shapes lying in π are independent over $\mathbb{Z}[q,q^{-1}]$. We now need to show that any q-codeterminant of shape $\lambda \in \pi$ can be expressed as a $\mathbb{Z}[q,q^{-1}]$-linear combination of standard q-codeterminants with shapes lying in π, but this is easily seen from Proposition 6.1.3 and the definition of an increasing saturation. ∎

6.2. Cellular algebras and quasi-hereditary algebras

It turns out that q-codeterminants are well-suited to describing the structure of the q-Schur algebra as a cellular algebra, in the sense of Graham and Lehrer. Using the Schur functor, we find that we can also describe the cellular structure of the Hecke algebras of type A.

We recall the definition of a cellular algebra from [GL].

Definition (Graham, Lehrer) Let R be a commutative ring with identity. A *cellular algebra* over R is an associative unital algebra, A, together with a cell datum $(\Lambda, M, C, *)$ where

1. Λ is a poset. For each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of tableaux of type λ) such that
$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A$$
is injective with image an R-basis of A.
2. If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear involutory anti-automorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.
3. If $\lambda \in \Lambda$ and $S, T \in M_\lambda$ then for all $a \in A$ we have
$$a.C^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \pmod{A(< \lambda)},$$
where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by the set
$$\{C^\mu_{S'',T''} : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.$$

Proposition 6.2.1 Let $R = \mathbb{Z}[q, q^{-1}]$. Let $\Lambda = \Lambda^+(n, r)$, ordered by reverse dominance so that $\lambda \leq \mu$ means $\lambda \trianglerighteq \mu$. Let $M(\lambda)$ be the set of standard tableaux of shape λ. Let C be the map taking the pair of elements $\langle S, T\rangle$ (where $S, T \in M(\lambda)$) to the associated standard q-codeterminant. Let $*$ be the anti-automorphism of $S_q(n, r)$ sending e_A to $e_{A^\top}$. Then $(\Lambda, M, C, *)$ is a cell datum for $A = S_q(n, r)$ over R.

Proof Axiom (1) follows from the fact (see Corollary 3.2.7) that the standard q-codeterminants give a free $\mathbb{Z}[q, q^{-1}]$-basis for A. It is easy to check from the definition of q-codeterminants that $C^\lambda_{S,T}$ is sent by $*$ (which is R-linear) to $C^\lambda_{T,S}$. Thus axiom (2) holds.

To show that axiom (3) holds, we use the techniques of the proof of Proposition 6.1.3. It is enough, by linearity, to check the axiom when $a = C^\nu_{S',T'}$ is a standard q-codeterminant. Consider the product of the two standard q-codeterminants given by

$$C^\nu_{S',T'} C^\lambda_{S,T} = (\xi_{s',\ell(\nu)}\xi_{\ell(\nu),t'}\xi_{s,\ell(\lambda)})\xi_{\ell(\lambda),t}.$$

Expanding the parentheses, we see from the strong straightening result that we can rewrite the whole expression as

$$\left(\sum_{\mu \trianglerighteq \lambda} a^\mu_{s'',t''} Y^\mu_{s'',t''}\right) \times \xi_{\ell(\lambda),t},$$

where the q-codeterminants $Y^\mu_{s'',t''}$ appearing are standard. This implies that if $\mu = \lambda$ then necessarily $t'' = \ell(\lambda)$. We deduce that

$$C^\nu_{S',T'} C^\lambda_{S.T} \equiv a^\lambda_{s'',\ell(\lambda)} Y^\lambda_{s'',t} \pmod{\trianglerighteq \lambda},$$

where $a^\lambda_{s'',\ell(\lambda)}$ is independent of t and T, and $Y^\lambda_{s'',t}$ is standard. This implies that axiom (3) holds. ∎

Now that we know that q-codeterminants describe the structure of q-Schur algebras as cellular algebras, we can use the results in [GL] (all of which have reasonably elementary proofs) to deduce some surprisingly strong results about q-Schur algebras. For example, we can easily find a complete set of irreducible modules over a field (which was first done in [DJ3]). We can also apply the *Schur functor* to obtain similar results about Hecke algebras of type A. It also turns out that, under some modest extra hypotheses which q-Schur algebras satisfy, cellular algebras are quasi-hereditary. This explains why q-codeterminants provide a good context in which to study the quasi-hereditary structure of q-Schur algebras.

The following proposition establishes that the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ can be regarded as a "cellular sub-algebra" of the q-Schur algebra, $S_q(n, r)$. We assume $n \geq r$, and we define $\omega \in \Lambda(n, r)^+$ to be

$(1, 1, \ldots, 1, 0, \ldots, 0)$, where there are r occurrences of the number 1. We know from [DJ3, §2] that $\mathcal{H}(\mathcal{S}_r)$ can be identified with the subalgebra of $S_q(n,r)$ given by $\xi_\omega S_q(n,r)\xi_\omega$.

Proposition 6.2.2 Suppose $n \geq r$. Let $R = \mathbb{Z}[q, q^{-1}]$. Let $\Lambda = \Lambda^+(n,r)$, ordered by reverse dominance so that $\lambda \leq \mu$ means $\lambda \trianglerighteq \mu$. Let $M'(\lambda)$ be the set of strongly standard tableaux of shape λ, i.e. those standard tableaux of shape λ which contain one occurrence of each of the numbers $1, 2, \ldots, r$. Let C' be the restriction of the map C given in Proposition 6.2.1 to the subalgebra $\mathcal{H}(\mathcal{S}_r)$. Let $*'$ be the restriction to $\mathcal{H}(\mathcal{S}_r)$ of the map $*$ appearing in Proposition 6.2.1. Then $(\Lambda, M', C', *')$ is a cell datum for $A = \mathcal{H}(\mathcal{S}_r)$ over R.

Proof A simple argument involving weight spaces shows that the Hecke algebra is spanned by the standard q-codeterminants it contains. It is now elementary to check that all the axioms hold. ∎

Remark

This result is interesting because one can now recover the representation theory of the symmetric group.

One of the striking features of cellular algebras is that their representation theory is easily accessible.

Definition (Graham, Lehrer) Let A be a cellular algebra with cell datum $(\Lambda, M, C, *)$. The left A-module $W(\lambda)$ is the free R-module with basis $\{C_S : S \in M_\lambda\}$ and A-action given by

$$a.C_S = \sum_{S' \in M(\lambda)} r_a(S', S) C_{S'},$$

where $a \in A$ and $S \in M(\lambda)$. The module W_λ affords the *cell representation* of A corresponding to λ.

Remark Axiom (3) ensures that this action is well-defined.

Proposition 6.2.3 When $A = S_q(n,r)$, the modules $W(\lambda)$ may be naturally identified with the q-Weyl modules of shape λ.

Proof Let B be the basic tableau of shape λ. The q-Weyl module, W_q^λ, which was defined in [DJ3, Definition 3.8 (ii)], has a highest weight vector, z_λ. We claim that the map $\rho : W_q^\lambda \to W(\lambda)$ which sends $s.z_\lambda$ to $s.C_B$, where $s \in A$, defines an isomorphism of A-modules.

It is enough to show that the action of standard q-codeterminants on both highest weight vectors is the same. By analysing the definition above and §5.1, we can describe this concisely.

In either case, a standard q-codeterminant of shape other than λ will annihilate either C_B or z_λ. Similar remarks apply to standard q-codeterminants of form $C^\lambda_{S,T}$ where $T \neq B$. The remaining basis elements clearly yield a basis of $W(\lambda)$ when acted on C_B, because $C^\lambda_{S,B}.C_B = C_S$, and they yield a similar basis for the q-Weyl modules by Theorem 5.3.1. This shows that ρ is well-defined and completes the proof. ∎

Graham and Lehrer [GL] introduce a certain bilinear form on $W(\lambda)$ which is instrumental in describing the irreducible modules.

Proposition 6.2.4 (Graham, Lehrer) Let A be a cellular algebra with cell datum $(\Lambda, M, C, *)$. The bilinear map $\phi_\lambda : W(\lambda) \times W(\lambda) \to R$ is defined by

$$C^\lambda_{S_1,T_1} C^\lambda_{S_2,T_2} \equiv \phi_\lambda(C_{T_1}, C_{S_2}) C^\lambda_{S_1,T_2} \pmod{A(< \lambda)},$$

where $S_1, T_1, S_2, T_2 \in M(\lambda)$. The function ϕ_λ is independent of T_1 and S_2, and satisfies the following equations for all $x, y \in W(\lambda)$ and for all $a \in A$:

$$\phi_\lambda(x, y) = \phi_\lambda(y, x), \tag{1}$$

$$\phi_\lambda(a^* x, y) = \phi_\lambda(x, ay). \tag{2}$$

Proof See [GL, Proposition 2.4]. ∎

We denote by Λ_0 the subset of $\lambda \in \Lambda$ satisfying $\phi_\lambda \neq 0$. We can now state

Theorem 6.2.5 (Graham, Lehrer) Let A be a cellular algebra with cell datum $(\Lambda, M, C, *)$ over a field R. Then the subspace of $W(\lambda)$ given by

$$\mathrm{rad}(\lambda) := \{x \in W(\lambda) : \phi_\lambda(x, y) = 0 \text{ for all } y \in W(\lambda)\}$$

is an A-submodule of $W(\lambda)$. Define $L(\lambda)$ to be $W(\lambda)/\mathrm{rad}(\lambda)$. Then the set $\{L(\lambda) : \lambda \in \Lambda_0\}$ is a complete set of (representatives of equivalence classes of) absolutely irreducible A-modules.

Proof See the results in [GL, §3]. ∎

Dipper and James [DJ3] define a certain bilinear form $\langle\langle\ ,\ \rangle\rangle$ on the q-Weyl modules. They show that it satisfies the properties (1) and (2) given in Proposition 6.2.4. We now show that the bilinear form ϕ_λ is essentially the same as $\langle\langle\ ,\ \rangle\rangle$.

Proposition 6.2.6 For each $\lambda \in \Lambda^+$ there exists an integer m such that $\phi_\lambda(x,y) = q^m\langle\langle x,y\rangle\rangle$ for all $x, y \in W(\lambda)$.

Proof We first prove that any bilinear form $\langle\ ,\ \rangle$ on the q-Weyl module $W(\lambda)$ satisfying properties (1) and (2) above must be a scalar multiple of the form ϕ_λ. Writing B as the basic tableau of shape λ, we now find that

$$\begin{aligned}\langle C^\lambda_{S,B} z_\lambda, C^\lambda_{T,B} z_\lambda\rangle &= \langle C^\lambda_{B,T} C^\lambda_{S,B} z_\lambda, z_\lambda\rangle \\ &= \phi_\lambda(C_T, C_S)\langle C^\lambda_{B,B} z_\lambda, z_\lambda\rangle \\ &= \phi_\lambda(C_T, C_S)\langle z_\lambda . z_\lambda\rangle\end{aligned}$$

The second line follows from the first by the fact that any standard q-codeterminant of shape other than λ annihilates the highest weight vector z_λ of the q-Weyl module. (See the proof of Proposition 6.2.3.) It was shown in [DJ3] that $\langle\langle z_\lambda, z_\lambda\rangle\rangle$ was equal to a certain integer power of q. The result follows because $\langle\langle\ ,\ \rangle\rangle$ satisfies properties (1) and (2). ∎

Examples The theory we have developed enables us to calculate explicitly the bilinear form ϕ_λ on the q-Weyl modules. Using Theorem 6.2.5, we can explicitly describe the irreducible modules for the q-Schur algebra, even in non-generic cases.

The Quantized Symmetric Powers

We now consider the quantized symmetric powers, in the sense of [GM]. These can be identified with the q-Weyl modules for $S_q(n,r)$ in the case where $\lambda = (r, 0, \ldots, 0)$. Let C_M and C_N be standard tableaux of shape λ (i.e. row increasing tableaux with one row). Let W_μ be the parabolic subgroup of $\mathcal{S}_r$ which is the row stabiliser of C_M, and let P_μ be its associated Poincaré polynomial. Let P_W be the Poincaré polynomial of $\mathcal{S}_r$. Using the product rule described in §2, we find that

$$\phi_\lambda(C_M, C_N) = \delta_{M,N}\frac{P_W}{P_\mu}(q).$$

Note that this quotient of polynomials becomes a multinomial coefficient as q is replaced by 1 (compare with [M, p. 83, Example 1]).

The Quantized Exterior Powers

Suppose now that $n \geq r$. We consider the quantized exterior powers, which can be identified with the q-Weyl modules for $S_q(n,r)$ in the case where $\lambda = \omega$. Let C_M and C_N be standard tableaux of shape λ (i.e. strictly column increasing tableaux with one column). Using the product rule described in §2, we find that

$$\phi_\lambda(C_M, C_N) = \delta_{M,N}.$$

This implies that the quantized exterior power is always absolutely irreducible when q is a unit.

One can obtain other interesting results for q-Schur algebras by using the theory of cellular algebras. Other results contained in [GL] include criteria for cellular algebras to be semisimple and quasi-hereditary.

As one would expect from corresponding results in the classical case (see [G2]) and the preceding discussion of cellular algebras, q-codeterminants are well suited to describe the structure of the q-Schur algebra as a quasi-hereditary algebra, in the sense of Cline, Parshall and Scott. With any quasi-hereditary algebra one associates a *defining sequence*, which is a certain chain of two-sided ideals of the algebra satisfying various technical conditions. (For the definitions and basic properties of quasi-hereditary algebras, the reader is referred to [M, §3.3].)

We recall the defining sequence for $S_q(n,r)$:

Theorem 6.2.7 (Parshall, Wang)

Let e_i be the idempotent of $S_q(n,r)$ given by $\xi_{\lambda^1} + \cdots + \xi_{\lambda^i}$, and define J_i to be the ideal Se_iS, where $S = S_q(n,r)$ and J_0 is taken to be 0. Then $S_q(n,r)$ is quasi-hereditary with weight poset Λ^+ and

$$\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t = S$$

is a defining sequence for S.

Proof See the remark in [Do3, §4]. ∎

We can restate this result in terms of q-codeterminants, thus obtaining a free $\mathbb{Z}[q,q^{-1}]$-basis of each J_i.

Corollary 6.2.8 The ideal J_i has as a free $\mathbb{Z}[q,q^{-1}]$-basis the set of all standard q-codeterminants of shapes $\lambda^1, \ldots, \lambda^i$, and is equal to the ideal $Y(\pi)$ corresponding to the increasing saturation $\pi = \{\lambda^1, \ldots, \lambda^i\}$.

Proof Since J_i is generated by e_i, we see it contains $\xi_{\lambda^s} = \xi_{\lambda^s} e_i \xi_{\lambda^s}$ where $s \leq i$. It is therefore generated as a two-sided ideal by the idempotents ξ_{λ^s} where $s \leq i$. By considering the natural basis of $S_q(n,r)$ we now find that J has as a spanning set the set of elements $\xi_{a,b}\xi_{\lambda^s}\xi_{c,d}$ where $s \leq i$, or, more economically, the set $\xi_{a,\ell(\lambda^s)}\xi_{\ell(\lambda^s),b}$ where again $s \leq i$. This is precisely the set of q-codeterminants of shape λ^s. Because $\succ$ refines $\triangleright$, the set $\{\lambda^1, \ldots, \lambda^s\}$, where $s \leq i$, is an increasing saturation, and the result now follows from Proposition 6.1.4. ∎

Before proceeding, it is worth noting two similarities between the basis of standard quantized codeterminants and Du's canonical (intersection cohomology) basis $\{\theta^d_{\lambda,\mu}\}$. This basis, which was introduced in [D2, §2], is in natural bijection with Dipper and James's basis $\{\phi^d_{\lambda,\mu}\}$. Like the basis of standard q-codeterminants, it has the property that bases for highest weight modules can be found by acting all the basis elements on the highest weight vector and taking the nonzero results (see [D3, Theorem 5.3] and Theorem 5.3.1). It is shown in [D3, Theorem 3.3] that the canonical basis also gives rise to the defining sequence for $S_q(n,r)$ in a natural way; again, each J_i is spanned by the basis elements it contains.

Du in [D1, Theorem 3.3] obtained (explicitly) the following result on quotients of Schur algebras, which is a special case of Theorem 6.3.5.

Proposition 6.2.9 (Du)

Let

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t = S(2,r)$$

be the defining sequence for the classical Schur algebra $S(2,r)$. Then

$$\frac{S(2,r)}{J_i} \cong S(2, r-2i).$$

Proof See [D1, Theorem 3.3]. ∎

We wish to find an analogue of this result in the quantized case for arbitrary n. It turns out that in general, quotienting out by ideals in the defining sequence does *not* produce smaller q-Schur algebras; the case $n = 2$ is a simple case because the order $\trianglerighteq$ happens to be total. It is however true that by quotienting by a suitable ideal corresponding to an increasing saturation, one can find an analogue of Proposition 6.2.9.

6.3. Epimorphisms between v-Schur algebras

It is convenient at this point to switch notation and to start working with the v-Schur algebra $S_v(n,r)$ and its associated v-codeterminants $\widehat{Y}^\lambda_{i,j}$ as defined in §4.2. We write $\widehat{Y}(\pi)$ for the ideal spanned by all v-codeterminants with shapes lying in π. Recall that $S_v(n,r)$ and $S_q(n,r)$ are related as rings via

$$S_v(n,r) \cong \mathcal{A} \otimes S_q(n,r)$$

and as $\mathbb{Z}[q,q^{-1}]$-modules via

$$S_v(n,r) \cong S_q(n,r) \oplus v.S_q(n,r).$$

We now aim to describe an epimorphism between certain v-Schur algebras and to investigate its properties.

In the main proof, we will need certain results in the representation theory of quantized enveloping algebras. We deal with these now.

Definition Define $L \in V^{\otimes n}$ to be the quantized exterior power (in the sense of §5.2) given by

$$\sum_{\sigma \in \mathcal{S}_n} (-v)^{-\ell(\sigma)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}.$$

Lemma 6.3.1 The vector L spans a 1-dimensional U module affording the representation which sends E_i and F_i to zero for all i, and K_j to $v.1$ for all j.

Proof The actions of E_i and F_i on L can be deduced from Lemma 5.2.7. The assertion for the K_j is clear by direct consideration of its action. ∎

Lemma 6.3.2 Let $b \in V^{\otimes r}$. Then the action of U on $V^{\otimes(r+n)}$ has the following properties.

1. If $u \in U^+$ then $u.(L \otimes b) = L \otimes (u.b)$.
2. If $u \in U^-$ then $u.(L \otimes b) = L \otimes (u.b)$.
3. If u is of the form
$$\begin{bmatrix} K_1;0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n;0 \\ t_n \end{bmatrix},$$
where $\sum_{i=1}^n t_i = n + r$, then $u.(L \otimes b) = L \otimes (u'.b)$, where u' is given by
$$\begin{bmatrix} K_1;0 \\ t_1 - 1 \end{bmatrix} \cdots \begin{bmatrix} K_n;0 \\ t_n - 1 \end{bmatrix},$$
or 0 if this does not make sense (because one or more of the t_i was zero).

Proof We first deal with the proof of part 1. We see from the nature of the identity that it is enough to check the statement for the algebra generators of U^+, i.e. the identity element, 1, and the elements E_i. The statement is clear for $u = 1$. If $u = E_i$, we find that u acts (via the coassociative comultiplication Δ) on $L \otimes b$ as

$$E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i.$$

The proof of part 1 now follows, because E_i annihilates L, as shown in Lemma 6.3.1.

The proof of part 2 follows similar lines to the proof of part 1. In this case, the action of F_i on $L \otimes b$ is given by acting

$$K_i^{-1} K_{i+1} \otimes F_i + F_i \otimes 1.$$

Again, we find that F_i annihilates L, so only the first term has any effect. One easily checks that $K_i^{-1} K_{i+1}$ acts as 1 on L, and part 2 now follows.

The proof of part 3 follows from Theorem 2.2.7 and Corollary 4.1.10. We see from those results that u will fix any basis vector of form $e_{i_1} \otimes \cdots \otimes e_{i_s} \in V^{\otimes(r+n)}$ where the multiplicity of e_j in the tensor is equal to t_j. Basis vectors not of this form are annihilated by u. The result follows because the multiplicity of each e_j in the element L is equal to 1. ∎

The following basic fact about standard tableaux turns out to be important in Theorem 6.3.5.

Lemma 6.3.3 Denote by $T(n,r)$ the set of standard tableaux with r boxes and entries from $\{1,\ldots,n\}$. There is an injection ι from $T(n,r)$ to $T(n,r+n)$ given by adding an extra column of length n and with entries $1,2,\ldots,n$ to the left of the tableau.

Proof This is immediate, because in a standard tableau, all entries in row i must be greater than or equal to i. This implies that the addition of the new column does not affect the property of standardness. It is clear that the new tableau is an element of $T(n,r+n)$. ∎

Example Suppose $n = 3$ and $r = 4$. Let T be the tableau given by

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\ .$$

Then $\iota(T)$ is given by

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\ .$$

Lemma 6.3.4 Let $[A][A']$ be a standard v-codeterminant corresponding to a pair of standard tableaux $\langle T, T' \rangle$. Then

$$A_{11} \geq A_{22} \geq \cdots \geq A_{nn} = A'_{nn} \leq \cdots \leq A'_{22} \leq A'_{11}.$$

Proof Note A_{ii} is the number of entries equal to i in row i of the tableau T; these entries must therefore occur at the left end of the row. Similarly, A'_{ii} is the number of entries equal to i in row i of T'. For any standard tableau, one sees that the number of entries equal to i in row i is greater than or equal to the number of entries equal to $i+1$ in row $i+1$. This implies that

$$A_{11} \geq \cdots \geq A_{nn}$$

and

$$A'_{11} \geq \cdots \geq A'_{nn}.$$

The fact that $A_{nn} = A'_{nn}$ follows from the requirement that T and T' must be of the same shape. We now find that A_{nn} and A'_{nn} are both expressions for the number of columns of length n in T (or T'). (This is because all entries in the n-th row are equal to n.) ∎

Definition

We define the linear map $\psi_{n,r} : S_v(n, r+n) \to S_v(n,r)$ by its effects on the standard v-codeterminants as follows:

$$\psi_{n,r}([A][A']) = \begin{cases} [A-I][A'-I] & \text{if } A_{nn} \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remarks

We see from Lemma 6.3.4 that this definition makes sense, and we see from Lemma 6.3.3 that it takes standard v-codeterminants in $S_v(n, r+n)$ to zero or to standard v-codeterminants in $S_v(n,r)$. Subtraction of the identity matrix (if possible) from each component of the standard v-codeterminant corresponds to removing the first column from each member of the corresponding pair of tableaux (if it is a column of length n).

Example Denote by $[T, T']$ the standard v-codeterminant corresponding to the ordered pair of standard tableaux $\langle T, T' \rangle$. Then

$$\left[\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \\ 3 & & \end{array} , \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{array} \right] \xrightarrow{\psi_{3,3}} \left[\begin{array}{cc} 1 & 2 \\ 2 & \end{array} , \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right]$$

and

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 3 \\ 2 & 3 & & \end{array} , \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & & \end{array} \right] \xrightarrow{\psi_{3,3}} 0.$$

We are now ready to state and prove an epimorphism theorem.

Theorem 6.3.5

1. The map $\psi_{n,r}$ is an epimorphism of $\mathbb{Z}[v, v^{-1}]$-algebras. Its kernel is the ideal $\widehat{Y}(\pi)$, where π is the set of $\lambda \in \Lambda^+$ such that a tableau of shape λ has no column of length n.
2. For a natural number k, define the linear map $\psi_{n,r,k} : S_v(n, r+kn) \to S_v(n,r)$ by

$$\psi_{n,r,k}([A][A']) = \begin{cases} [A-kI][A'-kI] & \text{if } A_{nn} \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $\psi_{n,r} = \psi_{n,r,1}$.)

The map $\psi_{n,r,k}$ is an epimorphism of algebras whose kernel is the ideal $\widehat{Y}(\pi_k)$, where π_k is the set of $\lambda \in \Lambda^+$ such that a tableau of shape λ has strictly fewer than k columns of length n.

3. The restriction of $\psi_{n,r,k}$ to $S_v^+(n, r+kn)$ (the Borel subalgebra of $S_v(n, r+kn)$) has the following effect on the basis elements $[A]$ of $S_v(n, r+kn)$:

$$\psi_{n,r,k}([A]) = \begin{cases} [A - kI] & \text{if there does not exist } i \text{ such that } A_{ii} < k, \\ 0 & \text{otherwise.} \end{cases}$$

The image of the restriction of $\psi_{n,r,k}$ is precisely $S_v^+(n,r)$, and $\ker \psi_{n,r,k}$ is spanned by the basis elements $[A]$ which it contains.

There is an analogous result for the Borel subalgebra S^-.

Proof

We first prove that ψ is an epimorphism. Surjectivity comes from Lemma 6.3.3, because the v-co-determinant in $S_v(n,r)$ corresponding to $\langle T, T' \rangle$ is the image under ψ of the v-codeterminant in $S_v(n, r+n)$ corresponding to $\langle \iota(T), \iota(T') \rangle$.

Next we define a certain map ψ from $S_v(n, r+n)$ to $S_v(n,r)$ which will turn out to coincide with $\psi_{n,r}$. Let $x \in S_v(n, r+n)$, and choose $u \in U$ such that $\theta_{r+n}(u) = x$ for the homomorphism $\theta_{r+n} : U(gl_n) \to S_v(n, r+n)$. (This means that u and x act in the same way on $V^{\otimes(r+n)}$, whichever u is chosen.) We now embed tensor space $V^{\otimes r}$ in tensor space $V^{\otimes(r+n)}$ via $b \mapsto L \otimes b$. The Hecke algebra $\mathcal{H}_r$ which acts on $V^{\otimes r}$ on the right may be naturally identified with a subalgebra of the Hecke algebra $\mathcal{H}_{r+n}$ acting on $V^{\otimes(r+n)}$ on the right by sending the generator $T_{(p,p+1)}$ to the generator $T_{(p+n,p+n+1)}$.

We define the linear map ψ to send $x \in S_v(n, r+n)$ to the endomorphism X of $V^{\otimes r}$ given by $u.(L \otimes b) = L \otimes X(b)$, where u is as above. The existence of X is a consequence of Lemma 6.3.2, as we shall explain shortly, and X is unique (i.e. independent of which u is chosen) because $S_v(n, r+n)$ is the faithful quotient of the action of U on $V^{\otimes(r+n)}$. It was shown in Theorem 4.2.5 that there exists such a u with the property that $u = u^- u^0 u^+$, where $u^- \in U^-$, $u^+ \in U^+$ and u^0 is of form

$$\begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix},$$

where $\sum_{i=1}^n t_i = r + n$. It now follows from Lemma 6.3.2 that X exists, since $X(b) = u^- u' u^+ .b$, where u^- and u^+ are as before, and u' is the element of U^0 given by

$$\begin{bmatrix} K_1; 0 \\ t_1 - 1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n - 1 \end{bmatrix},$$

or zero if this does not make sense. The embedding of $\mathcal{H}_r$ in $\mathcal{H}_{r+n}$ which we describe implies that the endomorphism of $V^{\otimes r}$ given by X commutes with the action of the associated Hecke algebra, and hence, by the double centralizer theorem, is equal to an element of $S_v(n,r)$. Thus ψ is well-defined.

To show that $\psi = \psi_{n,r}$, we recall Theorem 4.2.5. Choose $[A][A']$ to be a standard codeterminant of $S_v(n, r+n)$. We can rewrite this as $[A][D][A']$ for a suitable diagonal matrix D. (The (i,i)-entry of D is λ_i, where λ is the shape of the standard v-codeterminant concerned.) The Theorem exhibits an element u of the form we want which is naturally expressible as $u^- u^0 u^+$. The element u^- determines the entries below the diagonal of A (in a way described in Proposition 4.1.7), u^+ determines the entries above the diagonal of A', and

$$u^0 = \begin{bmatrix} K_1; 0 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ \lambda_n \end{bmatrix}.$$

Now consider the action of $u^- u^0 u^+$ on $L \otimes b$, where $b \in V^{\otimes r}$. We can now deduce several results from Lemma 6.3.2 and the expressions for $\theta_{r+n}(u^-)$ and $\theta_{r+n}(u^+)$ (given in Propositions 4.1.7 and 4.1.3):

a) The element u^0 acts as $[D - I]$ (or 0 if this does not make sense).

b) The element $u^- u^0$ acts as $[A - I]$ (or 0 if this does not make sense).

c) The element $u^0 u^+$ acts as $[A' - I]$ (or 0 if this does not make sense).

We know from the proof of Theorem 4.2.5 that u acts in the same way on $V^{\otimes(r+n)}$ as $u^- u^0 u^0 u^+$, so this implies that u acts as $[A - I][A' - I]$, or 0 this does not make sense. Hence the maps ψ and $\psi_{n,r}$ agree, and $\psi_{n,r}$ is an algebra homomorphism (because ψ clearly is, being defined in terms of an endomorphism algebra).

We postpone proof of the description of $\ker(\psi_{n,r})$ because it is a special case of $\ker(\psi_{n,r,k})$ for $k = 1$.

The proof that $\psi_{n,r,k}$ is an epimorphism of algebras follows similar lines to the corresponding proof in part 1, except that the embedding of $V^{\otimes r}$ in $V^{\otimes(r+kn)}$ is via

$$b \mapsto \underbrace{L \otimes \cdots \otimes L}_{k} \otimes b.$$

It was explained earlier why A_{nn} is the number of columns in the tableau of the same shape as the v-codeterminant, and because $\psi_{n,r,k}$ takes standard v-codeterminants to zero or to standard v-codeterminants, we see that $\ker \psi_{n,r,k}$ is as claimed.

We can prove part 3 by using the facts labelled above as a), b) and c). We concentrate on the case of $S_v^+(n, r + kn)$ because the case of $S_v^-(n, r + kn)$ is almost exactly the same. Recall that S^+ has as a basis all basis elements $[U]$ of $S_v(n, r + kn)$ such that U is an upper triangular matrix, and similarly S^- has as a basis all lower triangular basis elements $[L]$. We can now check from the preceding arguments that $\psi_{n,r,k}[A]$ is as claimed, when $[A]$ is lower or upper triangular. Note that

$\psi_{n,r,k}$ takes basis elements of $S_v^+(n, r + kn)$ to zero or to basis elements of $S_v^+(n, r)$, and that all triangular basis elements of $S_v^+(n, r)$ turn up in this way. This shows that the restricted image and restricted kernel of $\psi_{n,r,k}$ are as claimed, completing the proof of part 3.

Finally we prove that π and π_k are increasing saturations. Since $\pi = \pi_1$, we concentrate on the case for general k. Suppose a tableau of shape λ has fewer than k columns, and $\mu \rhd \lambda$. We need to show that μ has fewer than k columns. Saying that λ has fewer than k columns is equivalent to saying that $\lambda_n < k$. Since $\mu \rhd \lambda$, we find that in particular,

$$\lambda_1 + \cdots + \lambda_{n-1} \leq \mu_1 + \cdots + \mu_{n-1},$$

implying that $\mu_n \leq \lambda_n < k$ as required, since $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i = r$.

This completes the proof. ∎

Remarks Donkin [Do1] proves an analogue of this result in an even more general setting—that of *generalised Schur algebras*, although the analogue of the map ψ is not found explicitly. The map $\psi_{n,r}$ turns out [Do2] to be the dual of the coalgebra injection from $A(n, r)$ to $A(n, r + n)$ given by multiplication by the determinant function. (Here, $S(n, r)$ is the dual of $A(n, r)$.)

Notice that in the proof of Theorem 6.3.5, the natural basis for the Borel subalgebras plays the same rôle as the v-codeterminant basis for the v-Schur algebras. It should however be noted that the Borel subalgebras considered are not generalised Schur algebras [Do2].

6.4 Cellular inverse limits

Using the epimorphisms between v-Schur algebras studied in §6.3, we can define certain inverse limits of v-Schur algebras. It will turn out that the inverse limit has a natural cellular structure, and the quantized enveloping algebra $U(sl_n)$ embeds in it in a natural way.

First we note that the v-Schur algebra can be given a cellular structure which differs slightly from the one in §6.2.

Proposition 6.4.1 Let Λ, M and $\leq$ be as in Proposition 6.2.1. Let $R = \mathcal{A}$. Let C be the map taking the pair of elements $\langle S, T\rangle$ (where $S, T \in M(\lambda)$) to the associated standard v-codeterminant. Let $*$ be the anti-automorphism of $S_v(n, r)$ sending $[A]$ to $[A^\top]$ (i.e. transposition of basis elements). Then $(\Lambda, M, C, *)$ is a cell datum for $A = S_v(n, r)$ over R.

Proof The fact that $*$ is an anti-automorphism was proved in [BLM, Lemma 3.10]. The rest of the proof is analogous to the proof of Proposition 6.2.1. ∎

Definitions

It will be convenient to introduce the v-Schur algebra $S_v(n,r)$ for $r=0$. This is nothing other than the base ring $\mathcal{A}$, regarded as a quotient of $S_v(n,n)$ via the map $\psi_{n,0} : S_v(n,n) \to S_v(n,0)$. This map takes the standard v-codeterminant $[I][I]$ (where I is the identity matrix) to $1 \in \mathcal{A}$, and other standard v-codeterminants to 0.

Suppose $0 \leq r < n$ (where r is an integer as usual). We define $\widehat{S}_v(n,r)$ to be the set of semi-infinite sequences (x_i) satisfying

$$\cdots \xrightarrow{\psi_{n,r+3n}} x_3 \xrightarrow{\psi_{n,r+2n}} x_2 \xrightarrow{\psi_{n,r+n}} x_1 \xrightarrow{\psi_{n,r}} x_0,$$

where the element x_k lies in $S_v(n, r+kn)$.

Suppose r is as above. Define $\widehat{\Lambda}(n,r)$ to be the set

$$\{\lambda \in \Lambda^+(n, r+kn) \text{ for some } k \in \mathbb{N} \text{ with } \lambda_n = 0\}.$$

Lemma 6.4.2

Let λ lie in $\Lambda^+(n, r+kn)$ for some $k \geq 0$. Denote by ω the n-tuple $(1,1,\ldots,1)$. Then λ is of form $\lambda' + \lambda_n\omega$, where $\lambda' \in \widehat{\Lambda}$. Any standard v-codeterminant $[A][A'] \in S_v(n, r+kn)$ of shape λ satisfies

$$\psi_{n,r+(k-\lambda_n)n,\lambda_n}([A][A']) = [A - \lambda_n I][A' - \lambda_n I],$$

where the shape of $[A - \lambda_n I][A' - \lambda_n I]$ lies in $\widehat{\Lambda}(n,r)$.

If T, T' are elements of the same shape in $\widehat{\Lambda}(n,r) \cap \Lambda^+(n, r+(k+1)n)$ then $C(\langle T, T' \rangle)$ is a standard v-codeterminant of $S_v(n, r+(k+1)n)$ which is mapped to zero by $\psi_{n,r+kn}$.

Proof All the assertions are immediate from the definitions. ∎

Example Let $r=0$ and $n=3$. Let

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

and

$$T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}.$$

Then T and T' are both of shape $\lambda = (3,2,1)$ which lies in $\Lambda^+(3,6)$. This is of form $\lambda' + \omega$, where $\lambda' = (2,1,0)$. One easily sees that

$$\psi_{3,3,1}(C(\langle T, T' \rangle))$$

is a standard v-codeterminant of shape $\lambda' \in \widehat{\Lambda}(3,0) \cap \Lambda^+(3,3)$ which is mapped to zero by $\psi_{3,0}$.

We introduce a partial order, $\leq$, on the elements of $\widehat{\Lambda}(n,r)$ which is analogous to the dominance order on Λ^+.

Definition Let λ and μ be elements of $\widehat{\Lambda}(n,r)$. We define will write $\lambda \leq \mu$ if

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i + c.n$$

for some nonnegative integer c, and $\lambda \trianglerighteq \mu + c.\omega$.

Lemma 6.4.3 The order $\leq$ is a partial order on $\widehat{\Lambda}(n,r)$.

Proof This will follow from the fact that $\trianglerighteq$ is a partial order, and the observation that $\lambda \trianglerighteq \mu$ if and only if $\lambda + \omega \trianglerighteq \mu + \omega$ for $\lambda, \mu \in \Lambda^+(n,r)$.

Reflexivity is clear from the definition of $\leq$.

Let $\lambda, \mu \in \widehat{\Lambda}(n,r)$. Suppose $\lambda \leq \mu$ and $\mu \leq \lambda$. The definition of $\leq$ forces $c = 0$ in this case, and we have both $\lambda \trianglerighteq \mu$ and $\lambda \trianglelefteq \mu$. Since $\trianglerighteq$ is a partial order, we find that $\lambda = \mu$. This proves antisymmetry.

Suppose $\lambda, \mu, \nu \in \widehat{\Lambda}(n,r)$, $\lambda \leq \mu$ and $\mu \leq \nu$. We see from the definitions that

$$\lambda \trianglerighteq \mu + c_1.\omega$$

and

$$\mu \trianglerighteq \nu + c_2.\omega$$

for some natural numbers c_1 and c_2. Using the remark at the beginning of the proof, we find that

$$\mu + c_1\omega \trianglerighteq \nu + (c_1 + c_2)\omega,$$

and thus $\lambda \leq \nu$ by definition of $\leq$ and by transitivity of $\trianglerighteq$. Thus $\leq$ is transitive.

This proves that $\leq$ is a partial order. ∎

One can naturally define a set of standard v-codeterminants for $\widehat{S}_v(n,r)$ analogous to the standard v-codeterminants for the v-Schur algebras. If T and T' are standard tableaux of the same shape, we will write $C(T,T')$ for the associated standard v-codeterminant, using the map C which appears in Proposition 6.4.1. It should be noted that this set of v-codeterminants will not be a basis of $\widehat{S}_v(n,r)$: the set is linearly independent, but is not a spanning set.

Definition Let $C(T,T')$ be a standard v-codeterminant of $S_v(n, r+kn)$ of shape $\lambda \in \widehat{\Lambda}(n,r)$. Define the sequence (x_i) (where $x_i \in S_v(n, r+in)$) via

$$x_i := \begin{cases} 0 & \text{if } i < k, \\ C(T,T') & \text{if } i = k, \\ C(\iota^{i-k}(T), \iota^{i-k}(T')) & \text{if } i > k. \end{cases}$$

Here, the injection ι is as in §6.3. It follows from Theorem 6.3.5 and Lemma 6.4.2 that the sequence (x_i) defines an element of $\widehat{S}_v(n,r)$. We will denote this element by $\widehat{C}(T,T')$. Such elements will be referred to as standard v-codeterminants of $\widehat{S}_v(n,r)$.

In the results which follow, we will be considering certain "infinite sums"

$$\sum_{\langle T,T' \rangle} \widehat{a}(T,T')\widehat{C}(T,T').$$

Here, the elements $\widehat{a}(T,T')$ lie in $\mathcal{A}$, but there may be infinitely many v-codeterminants $C(T,T')$ occurring with nonzero coefficient. It to be understood that $\widehat{a}(T,T')$ is the total coefficient of $\widehat{C}(T,T')$ in the sum. ("Repeats" are not allowed.) This infinite sum is interpreted as an element of $\widehat{S}_v(n,r)$ as follows.

Consider a typical sum

$$\sum_{\langle T,T' \rangle} \widehat{a}(T,T')\,\widehat{C}(T,T'),$$

where the sum is infinite (as explained above) and the codeterminants appearing are standard. This is a sequence (x_i) (where $x_i \in S_v(n, r+in)$ as usual) as follows. With respect to the basis of standard v-codeterminants of $S_v(n, r+kn)$, the coefficient $a(S,S')$ of $C(S,S')$ in x_k is given by

$$a(S,S') = \begin{cases} \widehat{a}(T,T') & \text{if } (\iota^m(T), \iota^m(T')) = (S,S') \text{ for a nonnegative integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the integer m introduced above must be unique if it exists. Lemma 6.4.2 shows that this sequence (x_i) is indeed an element of $\widehat{S}_v(n,r)$.

Proposition 6.4.4 Each element of $\widehat{S}_v(n,r)$ can be expressed uniquely in the form

$$\sum_{\langle T,T' \rangle} \widehat{a}(T,T')\widehat{C}(T,T'),$$

where the elements $\widehat{a}(T,T')$ lie in $\mathcal{A}$ and the v-codeterminants $\widehat{C}(T,T')$ appearing are standard.

Proof Suppose we have a sequence (x_i) giving an element of $\widehat{S}_v(n,r)$. We define the coefficient $\widehat{a}(T,T')$ of a standard v-codeterminant $\widehat{C}(T,T')$ to be the coefficient $a(T,T')$ of $C(T,T')$ in x_k, where k is such that the tableaux T and T' have $r+kn$ boxes. Then we have

$$(x_i) = \sum_{\langle T,T' \rangle} \widehat{a}(T,T')\widehat{C}(T,T').$$

Thus any element of $\widehat{S}_v(n,r)$ is expressible in terms of an infinite sum of the required form. Moreover this expression is unique, using the above definition of the infinite sum. ∎

Lemma 6.4.5 The $\mathcal{A}$-module $\widehat{S}_v(n,r)$ is an associative algebra in the obvious way, i.e. if $x,\ y\ \in\ \widehat{S}_v(n,r)$ and $x = (x_i)$, $y = (y_i)$ then $xy := (x_i y_i)$.

Proof This works because the maps ψ are algebra homomorphisms. ∎

Note For our purposes, it is necessary to alter the axioms for a cellular algebra slightly. (For finite-dimensional algebras, this alteration makes no difference at all.) We now allow infinite R-linear combinations of basis elements $C^\lambda_{S,T}$ in the axioms. This alters the three axioms as follows.

1. We drop the hypothesis that the image of C is an R-basis for A. Instead, we ensure that each element of A can be written uniquely in the form

$$\sum a(S,T)\, C(S,T),$$

where $a(S,T) \in R$ is the coefficient of $C(S,T)$ in the sum. The sum may contain infinitely many elements $C(S,T)$ which occur with nonzero coefficients.

2. We strengthen the hypothesis that the anti-automorphism $*$ is R-linear by requiring it to satisfy

$$* : \sum a(S,T)\, C(S,T) \mapsto \sum a(S,T)\, C(T,S),$$

where the sums satisfy the same rules as above.

3. We extend the R-submodule $A(< \lambda)$ to consist of all elements

$$\sum a_{S'',T''} C^\mu_{S'',T''}$$

where $\mu < \lambda$, $S'' \in M(\mu)$ and $T'' \in M(\mu)$. The scalar $a_{S'',T''} \in R$ is the coefficient of $C^\mu_{S'',T''}$ in the sum. Again, infinitely many distinct $C^\mu_{S'',T''}$ may appear.

We call an algebra satisfying these modified axioms a *generalised cellular algebra*, and we call the quadruple $(\Lambda, M, C, *)$ a *generalised cell datum* for the algebra. Note that a generalised cellular algebra A over R satisfying the additional condition that the image of C is an R-basis for A is a cellular algebra in the usual sense. Also note that finite-dimensional generalised cellular algebras are the same as finite-dimensional cellular algebras.

Remarkably, many of the properties of cellular algebras studied in [GL] carry over to generalised cellular algebras, and the proofs are the same. For example, the finite-dimensional cell modules $W(\lambda)$

of §6.2 may be defined in the same way for generalised cellular algebras, and they have the properties one would expect.

We can now give a generalised cell datum for $\widehat{S}_v(n,r)$.

Proposition 6.4.6

Let $R = \mathcal{A}$. For $\lambda \in \widehat{\Lambda}(n,r)$, let $\widehat{M}(\lambda) = M(\lambda)$, where M is as in Proposition 6.2.1. Let $\widehat{C}$ be as above. Define the $\mathcal{A}$-linear map $\widehat{*} : \widehat{S}_v(n,r) \mapsto \widehat{S}_v(n,r)$ by

$$\widehat{*} : \sum \widehat{a}(T,T')\,\widehat{C}(T,T') \mapsto \widehat{a}(T,T')\,\widehat{C}(T',T).$$

Then $(\widehat{\Lambda}, \widehat{M}, \widehat{C}, \widehat{*})$ is a generalised cell datum for $\widehat{S}_v(n,r)$, where $\widehat{\Lambda}$ is partially ordered by $\leq$.

Proof Axiom (1) for generalised cellular algebras follows from Proposition 6.4.4 and the observation that $\widehat{M}(\lambda)$ is a finite set for each λ.

We must show that the map $\widehat{*}$ maps $\widehat{S}_v(n,r)$ to itself. Let (x_i) be an element of $\widehat{S}_v(n,r)$. Then $(\widehat{x_i^*})$ is another such sequence, because the maps ψ commute with the anti-automorphisms $*$ of the v-Schur algebras. (This is easy to see from Theorem 6.3.5.) The fact that it is an anti-automorphism follows from the definition of multiplication in $\widehat{S}_v(n,r)$. Axiom (2) now follows.

To check axiom (3), we choose a standard v-codeterminant, $\widehat{C}^{\lambda}_{T,T'}$. Define $k \in \mathbb{N}$ by stating that

$$\sum_{i=1}^{n} \lambda_i = r + kn,$$

where $0 \leq r < n$ is as in the statement of the proposition. Pick any element $x \in \widehat{S}_v(n,r)$. Suppose that $\widehat{C}^{\mu}_{S,S'}$ occurs with coefficient $r_x \neq 0$ in the expression for $x.\widehat{C}^{\lambda}_{T,T'}$. We know from Lemma 6.4.2 that $\widehat{C}^{\lambda}_{T,T'}$ is a sequence (y_i) where $y_i = 0$ if $i < k$. Because of the way the multiplication is defined, it must be the case that the sequence (z_i) corresponding to $\widehat{C}^{\mu}_{S,S'}$ also satisfies $z_i = 0$ for $i < k$. This implies that

$$\sum_{i=1}^{n} \mu_i = r + k'n$$

for some $k' \geq k$. This means that if $\mu \trianglerighteq \lambda + (k'-k).\omega$, we will have $\mu \leq \lambda$.

Consider the expression

$$x_{k'}.C(\iota^{k'-k}(T), \iota^{k'-k}(T'))$$

in $S_v(n, r + k'n)$. Theorem 6.3.5 shows that the coefficient of $C(S,S')$ in this expression is r_x. Proposition 6.4.1 now shows that $\mu \trianglerighteq \lambda + (k'-k).\omega$, and thus $\mu \leq \lambda$.

If $C(S,S')$ is of shape λ, the above technique and Proposition 6.4.1 show that r_x is independent of T'. (Note that $k = k'$ in this situation.)

Axiom (3) now follows. ∎

It is interesting to note that $\widehat{S}_v(n,r)$ has a natural topology on it which is compatible with the cellular structure.

Definitions For each nonnegative integer k, let $B_{k,r}$ be the subset of $\widehat{S}_v(n,r)$ given by

$$\{(x_i) : i < k \Rightarrow x_i = 0\}.$$

We stipulate that the set $\{B_{k,r} : k \in \mathbb{N}\}$ is a base of neighbourhoods of 0 in $\widehat{S}_v(n,r)$.

Proposition 6.4.7

(i) The algebra $\widehat{S}_v(n,r)$ has the structure of a Hausdorff topological ring in which the operation $\widehat{*}$ is a homeomorphism. The subsets $B_{k,r}$ are ideals of $\widehat{S}_v(n,r)$. With respect to this topology, $\widehat{S}_v(n,r)$ is complete.

(ii) As algebras,

$$\frac{\widehat{S}_v(n,r)}{B_{k+1,r}} \cong S_v(n, r+kn).$$

Proof We first prove part (i). The operation $+$ is clearly continuous by construction of the topology from the base. It is easy to check that the subsets $B_{k,r}$ are ideals of $\widehat{S}_v(n,r)$, which means that negation and multiplication are continuous. Clearly, $\widehat{*}$ maps $B_{k,r}$ to $B_{k,r}$, so it is continuous too (and hence a homeomorphism, since it is self-inverse). The Hausdorff property follows from the observation that

$$\bigcap_{i=0}^{\infty} B_{k,r} = \{0\}.$$

It is now not hard to see that every Cauchy sequence will converge, and thus that $\widehat{S}_v(n,r)$ is complete.

We now prove part (ii). The $\mathcal{A}$-linear map which projects the element (x_i) of $\widehat{S}_v(n,r)$ to x_k is an algebra homomorphism with kernel $B_{k+1,r}$. The proof follows. ∎

We now study the direct sum of all the algebras $\widehat{S}_v(n,r)$ for different values of r.

Definitions We define

$$\widehat{S}_v(n) := \bigoplus_{r=0}^{n-1} \widehat{S}_v(n,r).$$

This direct sum is a direct sum of $\mathcal{A}$-algebras. A base of neighbourhoods of zero is the set of subsets B_k as k ranges through $\mathbb{N}$ and

$$B_k := \bigoplus_{r=0}^{n-1} B_{k,r}.$$

Proposition 6.4.8 Let $R = \mathcal{A}$. Let $\Lambda^\circ = \Lambda^\circ(n)$ be the disjoint union $\bigcup_{r=0}^{n-1} \widehat{\Lambda}(n,r)$. (Note that Λ° is the set of all partitions of natural numbers into at most $n-1$ pieces.) We order Λ° via $\leq^\circ$. For $\lambda, \mu \in \Lambda^\circ$, we say $\lambda \leq^\circ \mu$ if λ and μ both lie in $\widehat{\Lambda}(n,r)$ for some r and $\lambda \leq \mu$ with respect to the order $\leq$ on $\widehat{\Lambda}(n,r)$.

For each $\lambda \in \Lambda^\circ$, let $M^\circ(\lambda)$ be the set of standard tableaux of shape λ. Let C° be the map taking a pair of standard tableaux $\langle T, T' \rangle$ (of the same shape $\lambda \in \Lambda^\circ$) to the element of $\widehat{S}_v(n)$ corresponding to the standard v-codeterminant $\widehat{C}^\lambda_{T,T'}$. Let $*^\circ$ be the anti-automorphism satisfying

$$*^\circ : \sum a^\circ(T,T')\, C^\circ(T,T') \mapsto \sum a^\circ(T,T')\ C^\circ(T',T).$$

Then $(\Lambda^\circ, M^\circ, C^\circ, *^\circ)$ is a generalised cell datum for $\widehat{S}_v(n)$.

Proof This is relatively easy to show after the work done in Proposition 6.4.6.

To verify axiom (1), we remark that $\leq^\circ$ is a partial order and $M^\circ(\lambda)$ is a finite set for each λ. The other assertions of the axiom follow from the direct sum decomposition in the definition of $\widehat{S}_v(n)$.

Axiom (2) is clear from the proof of Proposition 6.4.6 and the direct sum decomposition.

Axiom (3) follows quickly from the definition of $\leq^\circ$ and the direct sum decomposition. ∎

Proposition 6.4.9 The algebra $\widehat{S}_v(n)$ is a Hausdorff topological ring, where the operation $*^\circ$ is continuous. The subsets B_k are ideals of $\widehat{S}_v(n)$. With respect to this topology, $\widehat{S}_v(n)$ is complete.

Proof The proof is very similar to the proof of Proposition 6.4.7 (i). ∎

The motivation behind studying $\widehat{S}_v(n)$ is that we now show how to embed the quantized enveloping algebra $U(sl_n)$ in it. To do this, we will first work over the field $\mathbb{Q}(v)$ and embed $U(sl_n)$ in $\widehat{S}'(n) := \mathbb{Q}(v) \otimes \widehat{S}_v(n)$.

Following [D4], we identify $U(sl_n)$ with the subalgebra of $U(gl_n)$ generated by the elements

$$E_i,\ F_i,\ K_i K_{i+1}^{-1},\ K_i^{-1} K_{i+1},$$

as $1 \leq i < n$.

Definitions

Let $\widehat{S}'(n)$ be as above. We write $\widehat{S}'(n,r)$ for $\mathbb{Q}(v) \otimes \widehat{S}_v(n,r)$.

Let $u \in U(sl_n)$, and let $0 \leq r < n$.

The map $\widehat{\theta}_r : U \to \widehat{S}'(n,r)$ is defined to take u to the sequence $(\theta_{r+in}(u))$, where as usual θ_{r+in} is a certain epimorphism from $U(gl_n)$ to $S_v(n, r+in)$. (It remains to be seen that the resulting sequence is an element of $\widehat{S}'(n,r)$.)

The map $\widehat{\theta} : U \to \widehat{S}'(n)$ is defined to take u to

$$\bigoplus_{r=0}^{n-1} \widehat{\theta}_r(u).$$

Lemma 6.4.10

(i) For each $u \in U(sl_n)$ the sequence $(\theta_{r+in}(u))$ is an element of $\widehat{S}'(n,r)$.

(ii) The map $\widehat{\theta}_r : U \to \widehat{S}'(n,r)$ defined by $\widehat{\theta}_r(u) = (\theta_{r+in}(u))$ is an algebra homomorphism.

(iii) The map $\widehat{\theta} : U \to \widehat{S}'(n)$ defined by $\widehat{\theta}(u) = \bigoplus_{r=0}^{n-1} \widehat{\theta}_r(u)$ is an algebra homomorphism.

Proof It is clear that (ii) and (iii) will hold if the codomains are as claimed in the definitions, because all the maps θ_m are homomorphisms. Thus, the only nontrivial part of the proof is checking that (i) holds. Because all the maps θ_m are homomorphisms, it is enough to check this in the case where u is an algebra generator, i.e. , E_i, F_i, $K_i K_{i+1}^{-1}$ or $K_i^{-1} K_{i+1}$.

For each such element u, we need to check that

$$\psi_{n,r+in}(\theta_{r+(i+1)n}(u)) = \theta_{r+in}(u)$$

for each natural number i. This is an easy case-by-case check which uses part (3) of Theorem 6.3.5 and the definition of θ. We calculate explicitly the case of $K_i K_{i+1}^{-1}$ as an example.

We see from the definition of θ that

$$\theta_{r+(i+1)n}(K_i K_{i+1}^{-1}) = \sum_{D \in \mathbb{D}_{r+(i+1)n}} v^{d_i - d_{i+1}}[D].$$

Applying part (3) of Theorem 6.3.5, we find that $\psi_{n,r+(i+1)n}$ takes this element to

$$\sum_{D \in \mathbb{D}_{r+in}} v^{(d_i+1)-(d_{i+1}+1)}[D] = \sum_{D \in \mathbb{D}_{r+in}} v^{d_i - d_{i+1}}[D] = \theta_{r+in}(K_i K_{i+1}^{-1}).$$

This happens because $\psi_{n,r+(i+1)n}$ takes $[D]$ to $[D - I]$ if this makes sense, and to 0 otherwise.

The other cases are similar or easier. ∎

Remark This construction fails for $U(gl_n)$! Putting the element $K_i \in U(gl_n)$ through the check in the proof of Lemma 6.4.10 does not give the desired result.

The next aim is to prove that $\widehat{\theta}$ is injective.

Lemma 6.4.11 We have

$$\bigcap_{r=0}^{\infty} \ker \theta_r \cap U(sl_n) = 0.$$

Proof Suppose $u \in U = U(sl_n)$ lies in $\cap \ker \theta_r$ for all r. Then, because $S_v(n,r)$ is the quotient of $U(gl_n)$ which acts faithfully on tensor space $V^{\otimes r}$, u must annihilate $V^{\otimes r}$ for all values of r. It is known (see for example §5.1) that any finite-dimensional simple highest weight module L for $U(gl_n)$ (and hence for $U(sl_n)$) can be embedded in $V^{\otimes r}$ for suitable r. Thus u acts as zero on any finite-dimensional simple highest-weight module of $U(sl_n)$.

Lusztig [L3, Proposition 6.3.6] proves that an integrable U-module M is a sum of finite-dimensional simple highest weight U-modules. Therefore u annihilates M. We now see from [L3, Proposition 3.5.4] that $u = 0$, which completes the proof. ∎

Theorem 6.4.12 The map $\widehat{\theta}$ is a monomorphism of algebras from $U(sl_n)$ into $\widehat{S}_v(n)$. The generalised cell modules for $\widehat{S}_v(n)$ are in natural bijection with the simple highest weight modules for $U(sl_n)$.

Proof The fact that $\widehat{\theta}$ is a monomorphism of algebras follows from Lemmas 6.4.10 and 6.4.11.

The link between the two representation theories goes as follows.

Let $\mu = (\mu_1, \cdots, \mu_{n-1})$, where the elements μ_i are natural numbers. Define $\lambda_j = \sum_{i=j}^{n-1} \mu_i$ for each $1 \leq j < n$, and $\lambda_n = 0$. This means that $\mu_i = \lambda_i - \lambda_{i+1}$ and $\lambda \in \Lambda^\circ$. Set $r = \sum_{i=1}^{n} \lambda_i$.

The element μ is a dominant weight and can be identified with a typical simple highest weight module M for $U(sl_n)$. It can be lifted to a simple module M' for $U(gl_n)$ where the action of K_i on the highest weight vector z_λ (the same notation as in §5) is given by

$$K_i.z_\lambda = v^{\lambda_i}.z_\lambda.$$

We know from Theorem 5.3.1 and related results that M' can be identified with the v-Weyl module W_v^λ for $S_v(n,r)$, with r defined as above. Using a version of Proposition 6.2.3 suited to $S_v(n,r)$ rather than $S_q(n,r)$, we can identify this module with the cell module $W(\lambda)$. Using Proposition 6.4.7 (ii), one finds that this module corresponds to the generalised cell module of shape λ for $\widehat{S}_v(n,r')$, where r' and r are congruent modulo n.

Finally one can identify the module with the corresponding generalised cell module of shape λ for $\widehat{S}_v(n)$.

All the generalised cell modules of $\widehat{S}_v(n)$ turn up in this way. ∎

Definition Let $U_{\mathcal{A}}(sl_n)$ be the $\mathcal{A}$-subalgebra of $U(sl_n)$ generated by all $E_i^{(c)}$, $F_i^{(c)}$, $K_i K_{i+1}^{-1}$ and

$$\begin{bmatrix} K_i K_{i+1}^{-1}; 0 \\ c \end{bmatrix},$$

as $1 \leq i < n$ and $c \in \mathbb{N}$.

Lemma 6.4.13 The algebra $U_{\mathcal{A}}(sl_n)$ agrees with the $\mathcal{A}$-form of $U(sl_n)$, $U_{\mathcal{A}}^- \otimes {}'U^0 \otimes U_{\mathcal{A}}^+$, where ${}'U^0$ is as given in [D4, §2].

Proof The only nontrivial step is checking that

$$\begin{bmatrix} K_i K_{i+1}^{-1}; 0 \\ c \end{bmatrix}$$

lies in ${}'U^0$. Since the algebra given is an $\mathcal{A}$-algebra, it must be fixed setwise by the $\mathbb{Z}$-algebra homomorphism which sends v to v^{-1}. Applying this map to the given element changes it into

$$\begin{bmatrix} K_i^{-1} K_{i+1}; 0 \\ c \end{bmatrix},$$

which lies in ${}'U^0$. ∎

Corollary 6.4.14 The monomorphism $\widehat{\theta}$ restricts to an algebra monomorphism from $U_{\mathcal{A}}(sl_n)$ to $\widehat{S}_v(n)$.

Proof This follows from Lemma 6.4.13, which implies that if $u \in U_{\mathcal{A}}(sl_n)$ then $\theta_r(u) \in S_v(n,r)$. (As usual, $S_v(n,r)$ refers to our $\mathcal{A}$-form of the v-Schur algebra.) ∎

Finally, we look at the relationship between $\widehat{S}_v(n)$ and the algebra $\dot{U}$ of type A which was introduced in [L3, §23].

Definition Let $\dot{\Lambda}$ be the set of n-tuples of natural numbers $(\lambda_1, \ldots, \lambda_n)$ such that at least one of the λ_i is zero.

Lemma 6.4.15 There is a natural bijection between $\dot{\Lambda}$ and the set X_n of all $(n-1)$-tuples $\mu_1, \ldots, \mu_{n-1}$ of integers. This satisfies $\mu_i = \lambda_i - \lambda_{i+1}$.

Proof Clearly each λ determines a μ. Going the other way, we see that each μ clearly determines a λ up to adding an integer multiple of ω. The condition that all the λ_i are nonnegative and at least one is zero determines this multiple uniquely. ∎

Definitions To each $\lambda \in \dot{\Lambda}$ (corresponding as above to the $(n-1)$-tuple of integers μ), we associate an element e_μ of $\widehat{S}(n)$ as follows. Define $r' = \sum_{i=1}^n \lambda_i$, and let $r \equiv r' \pmod n$ be such that $0 \leq r < n$. Define k to be the natural number satisfying $r' = r + kn$. Let D be the diagonal matrix whose (i,i) entry is λ_i. Let (x_i) be the element of $\widehat{S}_v(n,r)$ given by setting

$$x_i := \begin{cases} 0 & \text{if } i < k, \\ [D + (i-k)I] & \text{otherwise.} \end{cases}$$

(It follows from part (3) of Theorem 6.3.5 that this is an element of $S_v(n,r)$.) We denote by e_μ the corresponding element of $\widehat{S}(n)$.

For each integer i such that $1 \leq i < n$ we define, following [L3, §2], the element $i' \in X_n$ (where X_n is as in Lemma 6.4.15) to be the i-th column of the Cartan matrix corresponding to $U(sl_n)$, namely

$$(i')_j := \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } |j-i| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.4.16 We regard $U(sl_n)$ as a subalgebra of $\widehat{S}_v(n)$ as in Theorem 6.4.12. The following relations hold in $\widehat{S}_v(n)$:

$$e_\mu e_{\mu'} = \delta_{\mu,\mu'} e_\mu \tag{1}$$

$$E_i e_\mu = e_{\mu+i'} E_i \tag{2}$$

$$F_i e_\mu = e_{\mu-i'} F_i \tag{3}$$

$$K_i K_{i+1}^{-1} e_\mu = v^{\mu_i} e_\mu \tag{4}$$

$$e_\mu K_i K_{i+1}^{-1} = v^{\mu_i} e_\mu \tag{5}$$

Proof All of these are easy consequences of the definitions and the multiplication in the v-Schur algebra. ∎

Definitions In [L3, §23], Lusztig defines a $\mathbb{Q}(v)$-algebra, $\dot{U}$, and proves that a basis for it is given by the set consisting of elements

$$b^- 1_\mu b^+,$$

as b^- ranges over a basis for U^-, b^+ ranges over a basis for U^+ and μ ranges over X_n. Lusztig proves that the following relations hold in $\dot{U}$:

$$1_\mu 1_{\mu'} = \delta_{\mu,\mu'} 1_\mu \tag{6}$$

$$E_i 1_\mu = 1_{\mu+i'} E_i \tag{7}$$

$$F_i 1_\mu = 1_{\mu-i'} F_i \tag{8}$$

$$K_i K_{i+1}^{-1} 1_\mu = v^{\mu_i} 1_\mu \tag{9}$$

$$1_\mu K_i K_{i+1}^{-1} = v^{\mu_i} 1_\mu \tag{10}$$

The multiplication in $\dot{U}$ is inherited in a natural way (explained in [L3, §23]) from the multiplication in $U(sl_n)$. It is not hard to see that the relations (6) to (10) together with the usual relations in $U(sl_n)$ are sufficient to express the product of two basis elements as a (finite) linear combination of others. Thus we can determine the structure constants of $\dot{U}$ with respect to the given basis.

The algebra $\dot{U}$ has an $\mathcal{A}$-form, $\dot{U}_\mathcal{A}$, which has as a free $\mathcal{A}$-basis the set $b^- 1_\mu b^+$ where b^- and b^+ range over $\mathcal{A}$-bases for $U_\mathcal{A}^-$ and $U_\mathcal{A}^+$ respectively, and μ ranges over X_n.

It will be convenient to reparametrise this basis by certain pairs of $n \times n$ matrices as follows.

We work with the basis B^- of U^- and B^+ of U^+ as in Proposition 1.5. This gives us a basis of $\dot{U}$ consisting of elements $b = b^- 1_\mu b^+$, where $b^- \in B^-$, $b^+ \in B^+$ and $\mu \in X_n$.

We define a *weakly positive* pair of matrices $\langle M, M' \rangle$ to be matrices with non-negative integer entries satisfying the following conditions:

(i) M is lower triangular;

(ii) M' is upper triangular;

(iii) the product $[M][M']$ is a v-codeterminant for some v-Schur algebra $S_v(n, r)$;

(iv) at least one of the matrices has a zero on the diagonal.

Lemma 6.4.17 There is a natural bijection between basis elements $b^- 1_\mu b^+$ of $\dot{U}$ and weakly positive pairs of matrices.

Proof Let $\langle M, M' \rangle$ be a weakly positive pair of matrices. Let λ be the n-tuple of natural numbers defined by setting λ_i to be the sum of the i-th column of M. (This is the same as the sum of the i-th row of M', by part (iii) of the definition of weakly positive.) Let m be such that $\lambda_m \leq \lambda_i$ for all i. Then $\lambda - \lambda_m.\omega$ lies in $\dot{\Lambda}$. Let $\mu \in X_n$ correspond to $\lambda - \lambda_m.\omega$ as in Lemma 6.4.15. We define b^- in B^- according to the entries below the diagonal in M, as in Proposition 4.1.7. We define b^+ according to the entries above the diagonal in M', as in Proposition 4.1.3. This produces a basis element $b^- 1_\mu b^+$ of $\dot{U}$.

We now describe the inverse of this process. Choose a basis element $b^- 1_\mu b^+$ of $\dot{U}$. Find the element $\lambda \in \dot{\Lambda}$ corresponding to μ as in Lemma 6.4.15. Let X be the lower triangular $n \times n$ matrix with integer entries as follows. The entries below the diagonal in X are determined by b^- as in Proposition 4.1.7. The diagonal entries may be negative, and are defined so that the sum of column i of X is λ_i. We also define X' to be the upper triangular $n \times n$ matrix with integer entries as follows. The entries above the diagonal are determined by b^+ as in Proposition 4.1.3. The diagonal entries (which may be negative) are chosen in the unique way which makes the sum of row i of X' equal to λ_i. Since $\lambda_i \in \dot{\Lambda}$, one of the matrices X or X' must have a nonpositive entry on the diagonal. Let m be the smallest such entry (possibly zero). Now define $M = X - m.I$ and $M' = X' - m.I$. It is easily checked that M is lower triangular, M' is upper triangular, the column sums of M are the row sums of M' (so $[M][M']$ is a v-codeterminant for sum $S_v(n, r)$) and one of M and M' has a zero entry on the diagonal. This therefore produces a weakly positive pair of matrices in a way which can be checked to be the inverse of the process in the preceding paragraph. ∎

We will now index the basis of $\dot{U}$ by weakly positive pairs of matrices $\langle M, M' \rangle$. The basis element corresponding to $\langle M, M' \rangle$ will be denoted by $\dot{C}(M, M')$

Definition Let $\langle M, M' \rangle$ be a weakly positive pair of matrices. We define an element $\widehat{Y}(M, M')$ of $\widehat{S}_v(n)$ as follows. Set r to be the sum of the entries of M (or equivalently of M'). Define r' such that $r' \equiv r \pmod{n}$ and $0 \leq r' < n$. Let k be such that $r = r' + kn$.

The element $\widehat{Y}(M, M')$ is the element of $\widehat{S}_v(n)$ identified with that element of $\widehat{S}_v(n, r')$ given by the sequence (x_i) where

$$x_i := \begin{cases} 0 & \text{if } i < k, \\ [M + (i-k)I][M' + (i-k)I] & \text{otherwise.} \end{cases}$$

One checks easily from Theorem 6.3.5 that this is an element of $\widehat{S}_v(n)$. (This works because of the triangular axioms for weakly positive pairs.)

It should be noted that all elements $\widehat{C}(T, T')$ turn up naturally in this way. It is not in general true that any such element will have the form $\widehat{C}(T, T')$, but if this is the case, we will call the basis element $\dot{C}(M, M')$ *standard*.

Before the proof of the main theorem, we require a certain technical result about classical Schur algebras.

Lemma 6.4.18 Let $[L][U]$ be a codeterminant for some (unquantized) Schur algebra $S(n,r)$, where L is a lower triangular matrix and U is an upper triangular matrix. Let w_L be the sum of all the entries below the diagonal of L, and let w_U be the sum of all the entries above the diagonal of U. Choose w to be greater than or equal to the maximum of w_L and w_U.

A unique basis element $[A]$ of $S(n, r+wn)$ exists with the following properties:

1. The entries below the diagonal of A are the same as those of L.
2. The entries above the diagonal of A are the same as those of U.
3. The column sums of A are the same as those of $U + w.I$.
4. The row sums of A are the same as those of $L + w.I$.

Consider the product $[L+w.I][U+w.I]$ in the Schur algebra $S(n, r+wn)$. Expanding the product in terms of the natural basis of $S(n, r+wn)$ (not the codeterminant basis), the basis element $[A]$ occurs with nonzero coefficient. All other basis elements occurring correspond to matrices of strictly larger trace. We call the element $[A]$ the *leading term* of the product $[L+w.I][U+w.I]$.

Proof Uniqueness of the matrix A is clear given existence. If we can show that a basis element $[A]$ occurring in the given product satisfies properties 1 and 2 then we have proved the first assertion, because properties 3 and 4 follow automatically from weight space considerations.

The fact that $[A]$ occurs with nonzero coefficient is a corollary of Schur's product rule and the choice of the number w, as is the fact that other basis elements occurring have larger trace. This is probably best illustrated with an example.

Let $n = r = 3$, let L be the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and let U be the matrix

$$\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, $w_L = 2$, $w_U = 3$. We choose $w = 3$. One can check that $[L][U]$ is nonzero by checking the row and column sums of each matrix. We now appeal to the tensor matrix space ideas of §2.1. The element of $T^{r+wn}(M_n)$ corresponding to $[L+w.I]$ consists of sums of permutations of the tensor

$$e_{11} \otimes e_{21} \otimes e_{21} \underbrace{\otimes e_{11} \otimes e_{22} \otimes e_{33}}_{w \text{ times}}$$

and the element corresponding to $[U + w.I]$ consists of sums of permutations of

$$e_{13} \otimes e_{13} \otimes e_{13} \underbrace{\otimes e_{11} \otimes e_{22} \otimes e_{33}}_{w \text{ times}}.$$

We have chosen w sufficiently large so that it is possible for two such permutations of tensors to multiply together (and give a nonzero result) so that no non-diagonal matrix unit in one combines with a non-diagonal matrix unit in the other. This gives rise to a basis element $[A]$ with properties 1 and 2. In this case, A is given by

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The other basis elements arise from two non-diagonal matrix units multiplying with each other. This gives rise to strictly more diagonal matrix units in the result, and a basis element of larger trace. ∎

Definition We define the $\mathcal{A}$-module $\widehat{\dot{U}}$ to be the set of all formal infinite sums

$$\sum c_{M,M'} \dot{C}(M, M'),$$

where the pairs $\langle M, M' \rangle$ are weakly positive and the element $c_{M,M'} \in \mathcal{A}$ is the coefficient of $\dot{C}(M, M')$ in the sum.

Theorem 6.4.19 Regard $U(sl_n)$ as a subalgebra of $\widehat{S}_v(n)$ as in Theorem 6.4.12.

(a) The $\mathcal{A}$-linear map $\dot{\theta}$ from $\dot{U}_{\mathcal{A}}$ to $\widehat{S}_v(n)$ which takes $\dot{C}(M, M')$ to $\widehat{Y}(M, M')$ is a homomorphism of $\mathcal{A}$-algebras whose image is dense with respect to the topology on $\widehat{S}_v(n)$. Furthermore, $\dot{\theta}$ is injective.

(b) The $\mathcal{A}$-module map $\dot{\theta}$ extends in a natural way to $\widehat{\dot{U}}$.

The kernel of $\dot{\theta}$ applied to $\widehat{\dot{U}}$ consists of infinite $\mathcal{A}$-combinations of elements of the form

$$\dot{C}(M, M') - \sum_{\langle A, A' \rangle} c_{A,A'} \dot{C}(A, A')$$

where $\dot{C}(M, M')$ is weakly positive but not standard, the elements $\dot{C}(A, A')$ are standard, and the coefficients $c_{A,A'}$ lie in $\mathcal{A}$.

Proof

We first prove part (a). We find from [L3, Lemma 23.2.2] and the relation (7) from earlier that $\dot{U}$ is generated as an algebra by the elements $1_\mu E_i^{(c)}$ and $F_i^{(c)} 1_\mu$, for all nonnegative integers c and elements $\mu \in X_n$.

Each element 1_μ in $\dot{U}$ is equal to an element $\dot{C}(D,D)$, where D is diagonal. It therefore maps to the idempotent $\widehat{Y}(D,D)$, which is equal to e_μ.

One checks that the element

$$1_\mu E_i^{(c)} = \dot{C}(D, D' + cE_{i,i+1})$$

maps under $\dot{\theta}$ to

$$e_\mu E_i^{(c)} = \widehat{Y}(D, D' + cE_{i,i+1}).$$

Here, D is such that

$$\dot{\theta}(1_\mu) = \dot{C}(D,D),$$

and as usual, $E_{i,i+1}$ is the matrix unit with 1 in the $(i,i+1)$-place and zeros elsewhere. The diagonal matrix D' is determined by the property that the sum of row h of $D' + cE_{i,i+1}$ is equal to D_{hh}. Analogous remarks hold for the element $F_i^{(c)} 1_\mu$.

We now have to check all the relations hold in the image. The usual relations of the quantized enveloping algebra certainly hold because of Theorem 6.4.12. The new relations (6) to (10) in $\dot{U}$ hold because of Lemma 6.4.16 and the similarity between the relations (1) to (5) in that lemma with the relations (6) to (10).

The "density" part of the argument holds because elements of $\widehat{S}_v(n)$ can be approximated arbitrarily "closely" (in the topology arising from the sets B_k) by finite $\mathcal{A}$-linear combinations of basis elements $\widehat{Y}(M, M')$.

We now need to prove that $\dot{\theta}$ is injective. Suppose to the contrary that there exists an element of $\dot{U}$ given by

$$u = \sum_{\langle M, M' \rangle} c_{M,M'} \dot{C}(M, M')$$

(where the scalars $c_{M,M'}$ are not all zero) which maps under $\dot{\theta}$ to zero. (Note that the sum is finite.) We may assume that this sum is as short as possible. Without loss of generality, we may assume all the $\dot{C}(M, M')$ are mapped by $\dot{\theta}$ into the same $\widehat{S}_v(n,r)$ for a fixed r. Also without loss of generality we may assume all the scalars $c_{M,M'}$ are elements of $\mathbb{Z}[v]$, and furthermore we may assume that they are not all divisible by $v - 1$. Thus by replacing v by 1, we obtain a counterexample to the corresponding unquantized problem. We may therefore concentrate on proving this.

With the aid of Lemma 6.4.18, we find that by taking a sufficiently large value of k, the k-th component of $\dot{\theta}(\dot{C}(M, M'))$ (i.e. that corresponding to $S(n, r + kn)$) contains (with nonzero coefficient) the leading term of the k-th component of the product corresponding to $\widehat{Y}(M, M')$. We can choose k so that this happens for every $\dot{C}(M,M')$ appearing in the sum u. (We have used the fact that the sum u is finite here. This is important.) Now consider the basis elements $\dot{C}(M,M')$ appearing which give rise to leading terms of minimal trace. By "minimal", we mean minimal among the set of traces of leading terms arising from elements $\dot{C}(M,M')$ appearing in u. One sees easily from the definition of leading term that different weakly positive pairs give rise to different leading terms. It is therefore not possible for any cancellation to occur among these leading terms of minimal trace, because Lemma 6.4.18 shows that other basis elements occurring have larger traces. We are forced to conclude that u is not in fact mapped to zero by $\dot{\theta}$, which proves the claim that $\dot{\theta}$ is injective.

We now tackle part (b). It is possible to extend $\dot{\theta}$ naturally to the infinite sums shown because for any open set B_k, all but finitely many basis elements $\dot{C}(M,M') \in \dot{U}$ map into B_k under $\dot{\theta}$. This means that the finite partial sums of an element in $\widehat{\dot{U}}$ will map under $\dot{\theta}$ to a Cauchy sequence in $\widehat{S}_v(n)$. Because of the completeness of $\widehat{S}_v(n)$, we can now extend $\dot{\theta}$ to $\widehat{\dot{U}}$.

We know $\dot{\theta} : \widehat{\dot{U}} \to \widehat{S}_v(n)$ is surjective because each standard v-codeterminant of $\widehat{S}_v(n)$ is the image of a standard basis element $\dot{C}(M,M')$.

Observe that the standard basis elements of $\dot{U} \subset \widehat{\dot{U}}$ map to elements $\widehat{C}(T,T')$ of $\widehat{S}_v(n)$. The non-standard basis elements map to elements which are images of elements $u \in \widehat{\dot{U}}$ expressible as infinite $\mathcal{A}$-combinations of standard elements $\dot{C}(A,A')$. Thus for any weakly positive non-standard pair $\langle M,M'\rangle$, one can find an infinite sum as (in the statement) lying in $\ker\dot{\theta}$. Note that $\dot{C}(M,M')$ and all the $\dot{C}(A,A')$ appearing map under $\dot{\theta}$ to elements in $\widehat{S}_v(n,r)$ for some fixed r.

Finally we note that infinite $\mathcal{A}$-combinations of elements as given in the statement are actually elements of $\widehat{\dot{U}}$. This happens because each standard basis element of $\dot{U}$, $\dot{C}(A,A')$, can only appear in finitely many infinite sums corresponding to different weakly positive pairs $\langle M,M'\rangle$ as in the statement of part (b). This follows from the fact that only finitely many elements $\dot{C}(M,M')$ lie outside any given open set B_k, and the observation that if $\dot{\theta}(\dot{C}(M,M'))$ lies in the set B_k, then so do all the elements $\dot{\theta}(\dot{C}(A,A'))$, where $\dot{C}(A,A')$ appears in the sum. (If the latter were not the case, one could use Proposition 6.4.6 (ii) to contradict the basis theorem for standard v-codeterminants.) The total coefficient of a standard element $\dot{C}(A,A')$ in any such infinite combination of infinite sums indexed by weakly positive pairs will therefore be a well-defined element of $\mathcal{A}$. (The analogous observation for non-standard elements $\dot{C}(M,M')$ is trivial because each such appears in only one of the infinite sums in the statement.) ∎

Concluding remarks

In [L3, §29], Lusztig essentially proves that $\dot{U}$ is cellular, and that its representation theory is the highest weight portion of the representation theory of $U(sl_n)$. This means that $\dot{U}$ and $\widehat{S}_v(n)$ have the same absolutely irreducible finite-dimensional modules.

By combining the results of Theorem 6.4.19 and Proposition 6.4.7 (ii) and using the direct sum decomposition of $\widehat{S}_v(n)$, one finds that the v-Schur algebras $S_v(n,r)$ are canonically quotients of the algebra $\dot{U}$. We can therefore reformulate the main results of §4 with $\dot{U}$ in the role of $U(gl_n)$. This seems to be a more natural situation than the one involving $U(gl_n)$: for example, one does not have the slightly messy problem of calculating $U^0 \cap \ker\theta$ which was dealt with in Proposition 4.1.12.

An application of the fact that $S_v(n,r)$ is a quotient of $\widehat{\dot{U}}$ (as an $\mathcal{A}$-module) is that all the coefficients $c_{A,A'}$ in the statement of Theorem 6.4.19 (b) are calculable, because they are the same as certain of the coefficients arising in the straightening formula for v-Schur algebras described in §3. To see this, take one of the expressions for an element of $\ker\dot{\theta}$ as described in Theorem 6.4.19 (b), and pass to the quotient $S_v(n,r)$. The nonstandard element of $\dot{U}$ occurring maps to a distinguished, but not standard, v-codeterminant (in the sense of §4) or to zero; the standard elements map to standard v-codeterminants, or to zero. For each nonzero element $c_{A,A'}$, one can choose a suitable r such that $\dot{C}(A,A')$ will not map to zero under the quotient map, but instead to $c_{A,A'}$ times a certain v-codeterminant. This means that our description of the elements of the kernel of the canonical map from $\widehat{\dot{U}}$ to $\widehat{S}_v(n)$ should be regarded as the limiting case of the straightening formula for distinguished v-codeterminants as described in Corollary 4.2.6.

One can also find results for $\dot{U}$ analogous to the main results in §5 dealing with the representation theory of $U(gl_n)$. Again, the proofs are more elegant.

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