Title: Positivity Properties for q-Schur Algebras

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POSITIVITY PROPERTIES FOR q-SCHUR ALGEBRAS

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Abstract. We prove Du's positivity conjecture for the canonical basis of the q-Schur algebra, using elementary arguments and the positivity result for Lusztig's canonical basis for $U_q^+(sl_n)$. We also describe a family of subalgebras of the q-Schur algebra, each of which is spanned by the canonical basis elements it contains.

0. Introduction

The q-Schur algebra, $S_q(n, r)$ was first introduced by Dipper and James in [**2**]. Du [**4**] introduced a *canonical basis* for this algebra, which is analogous to the bases of Kazhdan and Lusztig for Hecke algebras. Du conjectures [**7**, §5] that the structure constants with respect to this canonical basis are Laurent polynomials in the indeterminate $v := q^{\frac{1}{2}}$ with coefficients in $\mathbb{N}$. Du remarks that such a result ought to have a proof involving intersection cohomology, but that there should also be some algebraic or combinatorial approach to the problem. The main purpose of this paper is to prove Du's conjecture. The methods used are elementary in that they do not explicitly involve methods from intersection cohomology, but we shall require a deep result due to Lusztig, namely the positivity result for Lusztig's canonical basis of the quantized enveloping algebra of type A. Furthermore, the positivity result for q-Schur algebras is equivalent to Lusztig's result in the sense that an elementary proof of one would give an elementary proof of the other.

The rest of the paper falls into four parts, the first of which is expository.

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In §2, we study a certain class of subalgebras of $S_q(n,r)$, each of which turns out to be spanned by the canonical basis elements it contains.

In §3 we analyse a certain Lusztig–Dyer basis associated with the quantized enveloping algebra of type A. This turns out to be connected to the natural basis of the Borel subalgebra $S_q^+(n,r)$ of $S_q(n,r)$.

In §4, we prove the positivity result, first proving it for the Borel subalgebra, and then extending it to the whole algebra.

1. Du's Canonical Basis

In this paper, we shall denote the ring of Laurent polynomials $\mathbb{Z}[v,v^{-1}]$ by $\mathcal{A}$. We shall also be working with a certain $\mathcal{A}$-form of the q-Schur algebra, also denoted by $S_q(n,r)$, as no confusion is likely to arise. As always, the indeterminates q and v are related via $q=v^2$.

The canonical basis is a free $\mathcal{A}$-basis for $S_q(n,r)$. It first appeared in [**4**], where it was defined in the following way.

Let $W=\mathcal{S}_r$ be the symmetric group on r letters, and let S be the set of basic transpositions

$$(1,2),(2,3),\cdots,(r-1,r).$$

Let λ be a composition of r, namely

$$\lambda=(\lambda_1,\cdots,\lambda_m)$$

with $\lambda_1+\cdots+\lambda_m=r$ and $\lambda_i\geq 0$ for all i. Thus λ determines naturally a subset $I(\lambda)$ of S. We will write W_λ for the parabolic subgroup of W associated with this subset, and $\mathcal{D}_{\lambda,\mu}$ for the (unique) set of double coset representatives of minimal length corresponding to the double cosets $W_\lambda\backslash W/W_\mu$. We also write $\mathcal{D}^+_{\lambda,\mu}$ for the set of double coset representatives of maximal length.

We denote by $\mathcal{H}$ the Hecke algebra over $\mathcal{A}$ corresponding to W. This is a free $\mathcal{A}$-module with basis $\{T_w\}_{w\in W}$ satisfying the relations

$$T_sT_w=\begin{cases} T_{sw} & \text{if } \ell(sw)=\ell(w)+1,\\ qT_{sw}+(q-1)T_w & \text{otherwise,}\end{cases}$$

where $s \in S$ and $\ell(w)$ denotes the length of w. If D is a subset of W, we write

$$T_D := \sum_{x \in D} T_x.$$

We write x_λ for T_{W_λ}. Following [**3**], the q-Schur algebra is defined as

$$S_q(n,r) := \mathrm{End}_{\mathcal{H}} \left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H} \right).$$

(We are regarding $x_\lambda \mathcal{H}$ as a right module for $\mathcal{H}$ via right multiplication.) When q is replaced by 1, we recover the classical Schur algebra $S(n,r)$ which was described in [**9**].

For $\lambda, \mu \in \Lambda(n,r)$ and $w \in \mathcal{D}_{\lambda,\mu}$, we define the element $\phi^w_{\lambda,\mu} \in S_q(n,r)$ by

$$\phi^w_{\lambda,\mu}(x_\nu h) = \delta_{\mu,\nu} \sum_{x \in W_\lambda w W_\mu} T_x h,$$

where $\delta_{\mu,\nu}$ is the Kronecker delta. It was proved in [**3**] that the set

$$\{\phi^w_{\lambda,\mu} : \lambda, \mu \in \Lambda(n,r),\ w \in \mathcal{D}_{\lambda,\mu}\}$$

forms a basis of $S_q(n,r)$. We will call this the *natural basis* of $S_q(n,r)$.

We can define the canonical basis from this basis (using the same indexing set) by the formula (given in [**4**, (2.c)])

$$\theta^w_{\lambda\mu} = v^{\ell(w_{0,\mu})} \sum_{z \in \mathcal{D}_{\lambda,\mu}} \alpha_{z,w} \phi^z_{\lambda,\mu},$$

where $w_{0,\mu}$ is the longest element of W_μ and $\alpha_{z,w}$ is given by

$$v^{-\ell(w^+)} P_{z^+,w^+}.$$

Here the polynomial P is the usual Kazhdan–Lusztig polynomial, and we have denoted by w^+ the longest element in the same W_λ–W_μ double coset by w.

We also introduce the set $M(n,r)$ of $n \times n$ matrices with nonnegative integer coefficients whose sum is r. With each triple (λ, w, μ) where $\lambda, \mu \in \Lambda(n,r))$ and $w \in W$, one can associate an element of $M(n,r)$ in such a way that the triples indexing the natural basis are in bijection with the set $M(n,r)$. It is important for the purposes of §2 to understand this correspondence. Let $I(n,r)$ be the set of r-tuples with integer entries from 1 to n inclusive. If $\nu \in \Lambda(n,r)$, we write $\ell(\nu)$ for the element of $I(n,r)$ given by

$$(i_1, i_2, \dots, i_r),$$

where $i_1 \leq \cdots \leq i_r$ and s occurs ν_s times. Given a triple (λ, w, μ), we define the corresponding matrix $A \in M(n,r)$ by stipulating that $A_{i,j}$ is the multiplicity of the ordered pair $\langle i, j \rangle$ in the family

$$\{\langle \ell(\lambda)_{w(i)}, \ell(\mu)_i \rangle\},$$

as i ranges from 1 up to r. We will sometimes write θ_A for $\theta^w_{\lambda,\mu}$ in this case.

2. Good Subalgebras

Following [**10**, §2.3], we now define "good" subalgebras of the q-Schur algebra $S_q(n,r)$. Consider a set G of matrix units of M_n with the following two properties:

1. If $e_{ij} \in G$ and $e_{jk} \in G$ then $e_{ik} \in G$;
2. If $a > c$ and $b < d$, and if $e_{ab} \in G$ and $e_{cd} \in G$, then also $e_{ad} \in G$ and $e_{cb} \in G$.

Now consider the subset $G(n,r)$ of $M(n,r)$ defined by the condition

$$A \in G(n,r) \text{ iff } (A_{ij} \neq 0 \Rightarrow e_{ij} \in G).$$

Definition Let G be a good set of matrix units. We denote by $S_q(G)$ the subspace of $S_q(n,r)$ spanned by the set $\{\theta_A : A \in G(n,r)\}$.

Lemma 2.1 Let G be a good set of matrix units, and let $A \in G(n,r)$. Write $\theta^w_{\lambda,\mu}$ for θ_A (where w is any element of W), and let $w' \leq w$ in the (strong) Bruhat order. Let B be such that $\theta_B = \theta^{w'}_{\lambda,\mu}$. Then $B \in G(n,r)$.

Proof It was mentioned in [**8**, Lemma 2.2] that $w_B \leq w_A$ if and only if the sequence $(w_A(1), \dots, w_A(r))$ is obtainable from the sequence $(w_B(1), \dots, w_B(r))$ by a sequence of transpositions (i,j) where $i < j$ and i occurs to the left of j in $(w_B(1), \dots, w_B(r))$.

Now consider an arbitrary ascending sequence such as $\ell(\lambda)$ on which w_B and w_A act. (The case of the sequence $(1, 2, \ldots, r)$ was done in the last paragraph.) It follows by using the above techniques and omitting irrelevant transpositions that if $w_B \leq w_A$ then the sequence

$$(w_A(\ell(\lambda)_1), \ldots, w_A(\ell(\lambda)_r))$$

is obtainable from the sequence

$$(w_B(\ell(\lambda)_1), \ldots, w_B(\ell(\lambda)_r))$$

by a sequence of transpositions (i, j) where $i < j$ and i occurs to the left of j in $(w_B(\ell(\lambda)_1), \ldots, w_B(\ell(\lambda)_r))$.

Using the correspondence between the two forms of the basis, it is now not hard to see that the matrix B can be obtained from the matrix A in a sequence

$$(A = C_0, C_1, C_2, \ldots, C_m, C_{m+1} = B)$$

where each of the C_i lies in $M(n, r)$ and for each i, $C_{i+1} - C_i$ is a matrix of the form

$$e_{ab} + e_{cd} - e_{ad} - e_{cb},$$

where $a < c$ and $b < d$. Axiom 2 for a good set of matrix units now shows that all the matrices C_i lie in $G(n, r)$ (because e_{ab} and e_{cd} must be elements of G if e_{ad} and e_{cb} are); in particular, B does. $\square$

Theorem 2.2 The $\mathcal{A}$-subspace $S_q(G)$ of $S_q(n, r)$ is a subalgebra of $S_q(n, r)$ of dimension

$$\binom{m + r - 1}{r},$$

where m is the cardinality of G.

Proof This is a reworking of [**10**, Theorem 2.3.3], which says the same thing but which starts with a different definition of $S_q(G)$. It therefore suffices to prove that the two definitions are equivalent.

The definition of $S_q(G)$ in [**10**, §2.3] is the $\mathcal{A}$-module spanned by the natural basis elements $\{[A] : A \in G(n, r)\}$ for the good set G. (The element $[A]$ is a certain power of v times the element e_A.) Clearly this has the same dimension as the $S_q(G)$ we have just defined, so it is enough to show that if $A \in G(n,r)$ then θ_A is expressible as a linear combination of the set $\{[A'] : A' \in G(n,r)\}$. To prove this, we use Lemma 2.1 and the definition of the canonical basis.

By definition, we know that

$$\theta_{\lambda\mu}^{w^+} = v^{\ell(w_{0,\mu})} \sum_{z \in \mathcal{D}_{\lambda,\mu}} \alpha_{z,w} \phi_{\lambda,\mu}^{z^+},$$

where $w_{0,\mu}$ and $\alpha_{z,w}$ are as in §1. Because P_{z^+,w^+} (and hence $\alpha_{z,w}$) is only nonzero if $z^+ \leq w^+$, we can apply Lemma 2.1 to this expression and deduce that if the matrix indexing $\theta_{\lambda,\mu}^{w^+}$ is in $G(n,r)$, then so are all the matrices indexing the terms on the other side with respect to the natural basis. This proves the desired result. $\square$

Remarks

One can now deduce from the results about Zariski closures in the appendix of [**4**] that good subalgebras have a geometric interpretation in the approach of Beilinson, Lusztig and MacPherson [**1**]. They correspond to certain closed subalgebras of the q-Schur algebra with respect to the Zariski topology (i.e. subalgebras which which are unions of closed orbits, since there are only finitely many orbits) provided that $r > 1$. This follows because the ordering $\leq$ defined in [**1**] in terms of the Zariski closure is essentially the same as the strong Bruhat order.

It is also obvious that a positivity property for the canonical basis of any good subalgebra would follow from the positivity property of $S_q(n,r)$.

In §3 we will require the following result.

Corollary 2.3 The Borel subalgebra $S_q^+(n,r)$ spanned by all upper triangular elements $[U]$ is spanned by the canonical basis elements it contains, that is, the cones corresponding to upper triangular matrices.

Proof Note that the set $\{e_{ij} : i \leq j\}$ is good. For any θ_A, it will be true that θ_A lies in $S_q^+(n,r)$ if and only if all the terms in its expression in terms of the natural basis do. We now see from Theorem 2.2 that this will be true if and only if the matrix A is upper triangular. The assertion is now clear from dimension considerations. $\square$

3. Lusztig–Dyer Bases

The aim of §3 is to understand explicitly the connection between a certain Lusztig–Dyer basis of the quantized enveloping algebra $U^+(sl_n)$ and the natural basis of the q-Schur algebra $S_q(n,r)$. Once this is understood, we can compare Lusztig's canonical basis for $U^+(sl_n)$ with the canonical basis for $S_q(n,r)$. It will turn out that there is a close connection.

We then use this connection to deduce positivity properties for the Borel subalgebra $S_q^+(n,r)$.

The $\mathcal{A}$-subalgebra $U^+(sl_n)$ of the quantized enveloping algebra $U(sl_n)$ is generated by elements $E_1, \ldots, E_n$ subject to relations

$$E_i E_j = E_j E_i \quad \text{if } |i-j| > 1$$

and

$$E_i^2 E_j - (v+v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i-j| = 1.$$

The first aim (which is the hardest part of this section) is to show that a certain Lusztig–Dyer basis for $U^+(sl_n)$ agrees with the basis of $U^+(sl_n)$ which was studied in [**10**].

We denote by W the symmetric group $\mathcal{S}_r$ regarded as a Coxeter group of type A_{r-1}. The generators are the simple transpositions $s_p := (p, p+1)$.

We denote by $\alpha_{i,j+1}$ (where $i \leq j$) the positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$.

Let $s_{i_1} s_{i_2} \cdots s_{i_N}$ be a reduced expression for the longest word w_0 of W. Associated to this reduced expression is an ordering on the positive roots: the m-th root in the list is

$$s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m}),$$

where the action is the usual action of the Weyl group on the root system and α_{i_m} is the i_m-th fundamental root.

Lemma 3.1 The word

$$(s_n)(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_1\cdots s_n)$$

is a reduced expression for the longest element w_0 of W. It corresponds to the ordering on the positive roots of sl_n given as follows: the root $\alpha_{i,j}$ appears to the left of the root $\alpha_{k,l}$ if and only if $i > k$ or ($i = k$ and $j > l$).

Proof It is well-known (see for example [**15**]) that the given expression is a reduced expression for the longest element of W. The other assertion can be seen by a straightforward induction on n, using the fact that if $\alpha, \beta, \alpha+\beta$ are all positive roots, then the root $\alpha + \beta$ appears between the other two in the order. (This was mentioned in [**11**].) $\square$

3.2 The Lusztig–Dyer basis L corresponding to this reduced expression of the longest word was given in [**14**, 42.1.4] and consists of certain elements

$$E_{\alpha_1}^{(c_1)} E_{\alpha_2}^{(c_2)} \cdots E_{\alpha_N}^{(c_N)},$$

where N is the number of roots in the order $\alpha_1, \dots, \alpha_N$. The notation $E_\alpha^{(c)}$ means

$$\frac{E_\alpha^c}{[c]!},$$

where $[c] := [1][2] \cdots [c]$ and

$$[c] := \frac{v^c - v^{-c}}{v - v^{-1}}.$$

The basis studied in [**10**] for $U^+(sl_n)$ had a similar form, except that the corresponding elements $\widehat{E}_{\alpha_{i,j+1}}$ were defined recursively via $\widehat{E}_{\alpha_{i,i+1}} := E_i$ and

$$\widehat{E}_{\alpha_{i,j+1}} := \widehat{E}_{\alpha_{i,j}} \widehat{E}_{\alpha_{j,j+1}} - v^{-1} \widehat{E}_{\alpha_{j,j+1}} \widehat{E}_{\alpha_{i,j}}$$

for $|j - i| > 1$.

To prove the two bases agree, it is enough to prove that $E_{\alpha_{i,j}} = \widehat{E}_{\alpha_{i,j}}$ for all roots $\alpha_{i,j}$, because the orderings on the "root vectors" are the same in both cases. First, we will prove that they agree up to sign, and then we will prove that they agree. Since we are interested in positivity properties, it is vital that the signs work out correctly.

Lemma 3.3 For each positive root α_{ij},

$$E_{\alpha_{i,j}} = \pm \widehat{E}_{\alpha_{i,j}}.$$

Proof In [**12**, Lemma 2.5], Lusztig proves some identities about certain elements $E'_{\alpha_{i,j}}$ (which are denoted by $E_{\alpha_{i,j}}$, but we shall not do this, since they aren't the same as our elements $E_{\alpha_{i,j}}$). One can easily check that they satisfy

$$E'_{\alpha_{i,j+1}} = -E'_{\alpha_{i+1,j+1}} E'_{\alpha_{i,i+1}} + v^{-1} E'_{\alpha_{i,i+1}} E'_{\alpha_{i+1,j+1}}.$$

Lusztig shows [**12**, Corollary 4.3] that this element $E'_{\alpha_{i,j+1}}$ is equal to $T_{w_\alpha}(E_{i_\alpha})$, where T_{w_α} is a certain element of the braid group acting on $U(sl_n)$. Because $E'_{\alpha_{i,j+1}}$ lies in U^+, we invoke [**12**, 1.3(e)] and deduce that

$$T^{-1}_{w_\alpha^{-1}}(E_{i_\alpha}) = (T_{w_\alpha}(E_{i_\alpha}))^{\mathbf{op}},$$

where **op** is the opposition anti-automorphism of U.

If we denote the left hand side of this equality by $E''_{\alpha_{i,j+1}}$, then the above observation shows that

$$E''_{\alpha_{i,j+1}} = -E''_{\alpha_{i,i+1}} E''_{\alpha_{i+1,j+1}} + v^{-1} E''_{\alpha_{i+1,j+1}} E''_{\alpha_{i,i+1}}.$$

A routine induction on $j - i$ (using the fact that E_a and E_b commute when $|a - b| > 1$) shows that the elements $\widehat{E}$ satisfy the following identity:

$$\widehat{E}_{\alpha_{i,j+1}} = \widehat{E}_{\alpha_{i,i+1}} \widehat{E}_{\alpha_{i+1,j+1}} - v^{-1} \widehat{E}_{\alpha_{i+1,j+1}} \widehat{E}_{\alpha_{i,i+1}}.$$

Since $\widehat{E}_{i,i+1} = E''_{\alpha_{i,i+1}} = E_i$, we see by another easy induction on $|j - i|$ (using the above two equations) that

$$E''_{\alpha_{i,j+1}} = \pm \widehat{E}_{\alpha_{i,j}}.$$

We also find from the definition of the elements $E_{\alpha_{i,j+1}}$ appearing in the Lusztig–Dyer basis that

$$E_{\alpha_{i,j+1}} := \widetilde{T}_{w_\alpha}(E_{i_\alpha}) = \pm T^{-1}_{w_\alpha^{-1}}(E_{i_\alpha}) = \pm E''_{\alpha_{i,j+1}}.$$

The proof follows from these two observations. $\square$

To prove that our two bases are the same, it is now enough to work with the unquantized situation, as we are only interested in verifying that the signs are correct. This has the advantage of making the calculations much easier.

The braid group of type A_n is generated by elements $\widetilde{T}_1, \ldots, \widetilde{T}_{n-1}$ subject to the braid relations, namely

$$\widetilde{T}_i\widetilde{T}_j = \widetilde{T}_j\widetilde{T}_i$$

when $|i-j| > 1$ and

$$\widetilde{T}_i\widetilde{T}_j\widetilde{T}_i = \widetilde{T}_j\widetilde{T}_i\widetilde{T}_j$$

when $|i-j| = 1$. Lusztig [**14**, 37.1.3] defines an action of the braid group on $U(sl_n)$. In the case where $v = 1$ (i.e. the classical case), this proves the identities

$$\widetilde{T}_i(E_j) = -E_iE_j + E_jE_i$$

if $|i-j| = 1$ and

$$\widetilde{T}_i(E_j) = E_j$$

if $|i-j| > 1$.

Lemma 3.4 The following identities hold for $v = 1$:

$$\widetilde{T}_{j+1}(\widehat{E}_{\alpha_{i,j+1}}) = \widehat{E}_{\alpha_{i,j+2}}, \tag{1}$$

$$\widetilde{T}_{i-1}(\widehat{E}_{\alpha_{i,j+1}}) = -\widehat{E}_{\alpha_{i-1,j+1}}. \tag{2}$$

Proof To prove (1) and (2), we induct on $m = j - i$. The case $m = 0$ is done by the observations preceding the Lemma. Assume now that $m > 0$.

We first prove (1). Recall that

$$\widehat{E}_{\alpha_{i,j+1}} = \widehat{E}_{\alpha_{i,i+1}}\widehat{E}_{\alpha_{i+1,j+1}} - \widehat{E}_{\alpha_{i+1,j+1}}\widehat{E}_{\alpha_{i,i+1}}.$$

Lusztig shows that the action of an element of the braid group is an algebra homomorphism. Since $m > 0$, we see that $\widetilde{T}_{j+1}(\widehat{E}_{\alpha_{i,i+1}}) = \widehat{E}_{\alpha_{i,i+1}}$. Using the inductive hypothesis, we find that

$$\widetilde{T}_{j+1}(\widehat{E}_{\alpha_{i,j+1}}) = \widehat{E}_{\alpha_{i,i+1}}\widehat{E}_{\alpha_{i+1,j+2}} - \widehat{E}_{\alpha_{i+1,j+2}}\widehat{E}_{\alpha_{i,i+1}}.$$

The identity (1) now follows.

The proof of (2) is similar but uses the identity

$$\widehat{E}_{\alpha_{i,j+1}} = \widehat{E}_{\alpha_{i,j}}\widehat{E}_{\alpha_{j,j+1}} - \widehat{E}_{\alpha_{j,j+1}}\widehat{E}_{\alpha_{i,j}}. \quad \square$$

Lemma 3.5 Let $v = 1$. Then

$$E_{\alpha_{1,n}} = \widehat{E}_{\alpha_{1,n}}.$$

Proof Let $l = n - 1$. The definition of $E_{\alpha_{1,n}}$ is

$$(\widetilde{T}_l)(\widetilde{T}_{l-1}\widetilde{T}_l)(\widetilde{T}_{l-2}\widetilde{T}_{l-1}\widetilde{T}_l)\cdots(\widetilde{T}_2\cdots\widetilde{T}_l)(E_1).$$

By repeatedly discarding irrelevant $\widetilde{T}$s on the right and commuting other $\widetilde{T}$s as far to the right as possible, one sees that this is equal to

$$\widetilde{T}_l\widetilde{T}_{l-1}\cdots\widetilde{T}_2(E_1).$$

Repeated application of Lemma 3.4 (1) now shows that this is equal to $\widehat{E}_{\alpha_{1,n}}$, as required. □

Lemma 3.6 Let $v = 1$. Then

$$E_{\alpha_1} = \widehat{E}_{\alpha_1}.$$

Proof The right hand side is equal to E_1 by definition. Letting $l = n - 1$, the left hand side is defined as

$$(\widetilde{T}_l)(\widetilde{T}_{l-1}\widetilde{T}_l)\cdots(\widetilde{T}_2\cdots\widetilde{T}_l)(\widetilde{T}_1\cdots\widetilde{T}_{l-1})(E_l).$$

Because the last root in the list is known to be α_l, and therefore

$$(s_l)(s_{l-1}s_l)\cdots(s_2\cdots s_l)(s_1\cdots s_{l-1})(\alpha_l) = \alpha_1,$$

the result follows from [**13**, 1.3(c)]. □

Lemma 3.7 Let $v = 1$, and suppose $1 < a < n$ for $a \in \mathbb{N}$. Then

$$(\widetilde{T}_2\widetilde{T}_3\cdots\widetilde{T}_{n-1})(\widetilde{T}_1\widetilde{T}_2\cdots\widetilde{T}_{a-1})(E_a) = -\widetilde{T}_1\widetilde{T}_2\cdots\widetilde{T}_a(E_{a+1}).$$

Proof We first claim that if $2 \leq m \leq a$ then

$$T' = (\widetilde{T}_2\widetilde{T}_3\cdots\widetilde{T}_m)(\widetilde{T}_1\widetilde{T}_2\cdots\widetilde{T}_a)(E_{a+1}) = (\widetilde{T}_2\widetilde{T}_3\cdots\widetilde{T}_{m-1})(\widetilde{T}_1\widetilde{T}_2\cdots\widetilde{T}_a)(E_{a+1}).$$

By commuting the $\widetilde{T}_m$ to the right as far as possible, we find that

$$T' = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_{m-1})(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_m\widetilde{T}_{m-1}\widetilde{T}_m\widetilde{T}_{m+1} \cdots \widetilde{T}_a)(E_{a+1})$$

Now rewrite $\widetilde{T}_m\widetilde{T}_{m-1}\widetilde{T}_m = \widetilde{T}_{m-1}\widetilde{T}_m\widetilde{T}_{m-1}$ to get

$$T' = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_{m-1})(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_{m-1}\widetilde{T}_m\widetilde{T}_{m-1}\widetilde{T}_{m+1} \cdots \widetilde{T}_a)(E_{a+1}).$$

We now commute the $\widetilde{T}_{m-1}$ all the way to the right, where it acts trivially on E_{a+1}. It can therefore be discarded. Thus

$$T' = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_{m-1})(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_{m-1}\widetilde{T}_m\widetilde{T}_{m+1} \cdots \widetilde{T}_a)(E_{a+1}).$$

This establishes the claim.

Now let

$$T = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_{n-1})(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_{a-1})(E_a).$$

By commuting all $\widetilde{T}_i$ where $i > a+1$ to the right, we note that

$$T = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_{a+1})(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_{a-1})(E_a).$$

Then we commute the $\widetilde{T}_{a+1}$ to the right to give

$$T = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_a)(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_{a-1})\widetilde{T}_{a+1}(E_a).$$

Now rewrite $\widetilde{T}_{a+1}(E_a)$ as $-\widetilde{T}_a(E_{a+1})$. Now we have

$$-T = (\widetilde{T}_2\widetilde{T}_3 \cdots \widetilde{T}_a)(\widetilde{T}_1\widetilde{T}_2 \cdots \widetilde{T}_a)(E_{a+1}).$$

Repeated applications of the initial claim now yield the desired result. $\square$

Definitions

Let Ψ^+ be the algebra homomorphism from the subalgebra of $U(sl_n)$ generated by

$$\{E_i, F_i, K_i, K_i^{-1} : i < n-1\}$$

to the subalgebra of U generated by

$$\{E_i, F_i, K_i, K_i^{-1} : i > 1\}$$

which takes E_i to E_{i+1}, F_i to F_{i+1}, K_i to K_{i+1} and K_i^{-1} to K_{i+1}^{-1}.

Denote by Ψ^- be the inverse to this map. It is clear from the nature of the relations of $U(sl_n)$ that Ψ^+ and Ψ^- are algebra homomorphisms.

Lemma 3.8 Suppose $i > 1$ and $j < n$. Then $\Psi^-(\widehat{E}_{i,j}) = \widehat{E}_{i-1,j-1}$ and $\Psi^+(\widehat{E}_{i,j}) = \widehat{E}_{i+1,j+1}$.

Proof The proof in both cases is a straightforward induction on $|j-i|$. $\square$

Proposition 3.9 For all positive roots $\alpha_{i,j}$ with $1 \leq i < j \leq n$,

$$E_{\alpha_{i,j}} = \widehat{E}_{\alpha_{i,j}}.$$

Proof By the remarks following Lemma 3.3, we need only verify the statement of the Proposition for the case $v = 1$. The proof is by induction on n. The base case of $n = 2$ is obvious.

Now consider all the roots except the last $(n-1)$ in the list. We see that $\alpha_{i,j}$ is such a root if and only if $i > 1$. The definition of the corresponding $E_{\alpha_{i,j}}$ is of the form

$$\widetilde{T}_{i_1} \cdots \widetilde{T}_{i_{m-1}}(E_{i_m}),$$

where all the entries i_s appearing are strictly greater than 1. (Note that in our reduced expression for w_0, there is only one occurrence of s_1.)

Now consider

$$E' = \widetilde{T}_{i_1-1} \cdots \widetilde{T}_{i_{m-1}-1}(E_{i_m-1}).$$

The nature of Lusztig's action of the braid group shows that this is in fact equal to $\Psi^-(E_{\alpha_{i,j}})$. However, we see by reference to the case of $(n-1)$ that $E' = E_{\alpha_{i-1,j-1}}$. Hence

$$E_{\alpha_{i,j}} = \Psi^+(E') = \Psi^+(E_{\alpha_{i-1,j-1}}).$$

We now use the inductive hypothesis (for the case $(n-1)$) and Lemma 3.8 to deduce that

$$\Psi^+(E_{\alpha_{i-1,j-1}}) = \Psi^+(\widehat{E}_{\alpha_{i-1,j-1}}) = \widehat{E}_{\alpha_{i,j}}$$

as required.

We now turn our attention to the last $(n-1)$ roots in the list. The longest root is dealt with by Lemma 3.5. The last root is dealt with by Lemma 3.6. The other root vectors are of the form $E_{1,a}$ for $2 < a < n$. Denote by $E_{i,j,n-1}$ the Lusztig–Dyer basis element corresponding to $\alpha_{i,j}$ in the case $n-1$ (rather than n).

Working from the definition of $E_{1,a}$, applying Lemma 3.7 and commuting the $\widetilde{T}_1$ appearing to the left, we find that

$$E_{1,a} = -\widetilde{T}_1(\Psi^+(E_{1,a-1,n-1})).$$

Now by using the inductive hypothesis on n, Lemma 3.8 and Lemma 3.4 (2), we find that

$$E_{1.a} = -\widetilde{T}_1(\Psi^+(\widehat{E}_{1,a-1})) = -\widetilde{T}_1(\widehat{E}_{2,a}) = \widehat{E}_{1,a}.$$

This completes the proof. $\square$

The reason for going to so much trouble in checking that the two bases of U^+ agree is that the Lusztig–Dyer basis that we have been studying is closely related to the natural basis $\{[A] : A \in M(n,r)\}$ of $S_q(n,r)$. In [**6**], a certain algebra homomorphism $\pi_r : U(sl_n) \to S_q(n,r)$ was introduced. We use this to relate the Lusztig–Dyer basis to a basis of $S_q^+(n,r)$, the Borel subalgebra of $S_q(n,r)$ spanned by all upper triangular basis elements.

Proposition 3.10 Let X be the set of orthogonal idempotents $\{[D]\}$ of $S_q(n,r)$ parametrised by diagonal matrices in $M(n,r)$. Denote by B the Lusztig–Dyer basis of $U(sl_n)$, as defined in §3.2. Then the nonzero elements of the set

$$\{[D].\pi_r(b) : [D] \in X, b \in B\}$$

are precisely the elements

$$\{[U] \in S_q(n,r) : U \text{ is upper triangular}\}.$$

Proof This follows from [**10**, Propositions 4.1.3, 4.2.1], because we know by Proposition 3.9 that the two bases are equivalent. $\square$

4. The Positivity Result

The next aim is to use Proposition 3.10 to establish the positivity property for $S_q^+(n,r)$.

Definitions Following [**4**], we write $\widehat{\phi}^w_{\lambda,\mu}$ for $v^{\ell(w^+)+\ell(w_{0,\mu})}\phi^w_{\lambda,\mu}$. Note that if $\phi^w_{\lambda,\mu}$ corresponds to $A \in M(n,r)$, then $\widehat{\phi}^w_{\lambda,\mu} = [A]$.

In [**4**, §3.1], a $\mathbb{Z}$-linear involution, $\bar{\ }$, on $S_q(n,r)$ is described. This sends v to v^{-1} and fixes all $\theta^w_{\lambda,\mu}$.

The algebra U^+ also has a $\mathbb{Z}$-linear automorphism, $\bar{\ }$, sending v to v^{-1} and fixing all the E_i. (No confusion will arise in the similarity of notation as will be seen shortly.)

Lemma 4.1

(a) For all possible λ and μ, the identity

$$\theta^1_{\lambda,\mu} = \widehat{\phi}^1_{\lambda,\mu}$$

holds in $S_q(n,r)$.

(b) For all $1 \leq i < n$, $\overline{\pi_r(E_i)} = \pi_r(E_i)$.

(c) Let $u \in U^+$. Then $\overline{\pi_r(u)} = \pi_r(\bar{u})$. Thus if $u \in U$ satisfies $\bar{u} = u$ then $\overline{\pi_r(u)} = \pi_r(u)$.

Proof The proof of (a) follows from the expression for the θ basis in terms of the $\widehat{\phi}$ basis given in [**4**, §3.8], and the fact that if $z^+ \leq 1^+$ in the Bruhat order then the distinguished double coset representative of z is 1.

The proof of (b) follows using the definition (as in [**6**]) of $\pi_r(E_i)$ which is of the form

$$\sum_{\langle\lambda,\mu\rangle} \widehat{\phi}^1_{\lambda,\mu}$$

for a certain set of pairs $\langle\lambda,\mu\rangle$.

For (c), observe that (b) shows it is true for the generators E_i. Because $\bar{\ }$ is an $\mathcal{A}$-semilinear algebra homomorphism and π_r is an $\mathcal{A}$-linear algebra homomorphism, the result follows. $\square$

Definitions We define $\mathcal{A}^+$ to be $\mathbb{Z}[v]$, and $\mathcal{A}^-$ to be $\mathbb{Z}[v^{-1}]$.

Denote by $\mathcal{L}$ the free $\mathcal{A}^-$-module with the Lusztig–Dyer basis L as a basis. There is a canonical projection

$$\pi_{\mathcal{L}} : \mathcal{L} \rightarrow \mathcal{L}/v^{-1}\mathcal{L}.$$

Denote by $\mathcal{L}'$ the free $\mathcal{A}^-$-module with $\{[U] : U \text{ is upper triangular}\}$ as a basis. There is a canonical projection

$$\pi_{\mathcal{L}'} : \mathcal{L}' \rightarrow \mathcal{L}/v^{-1}\mathcal{L}'.$$

Following [**14**, Theorem 42.1.10, Corollary 42.1.13], we can characterise the *canonical basis* B of $U^+(sl_n)$ by the following two properties.

1. For each $b \in B$, $\bar{b} = b$.
2. For each $b \in B$, $\pi_{\mathcal{L}}(b) = \pi_{\mathcal{L}}(b')$ for exactly one $b' \in L$.

Using Corollary 2.3 and [**4**, Theorem 3.9], we can characterise the canonical basis B' of $S_q^+(n, r)$ by the properties

$1'$. For each $b \in B'$, $\bar{b} = b$.

$2'$. For each $b \in B'$, $\pi_{\mathcal{L}'}(b) = \pi_{\mathcal{L}'}(b')$ for exactly one $b' = [U]$ where U is upper triangular.

Proposition 4.2 The nonzero elements

$$\{[D].\pi_r(b) : b \in B, D \text{ diagonal}, D \in M(n, r)\}$$

are precisely the canonical basis elements θ_U in $S_q(n, r)$.

Proof We need to verify properties $1'$ and $2'$ above. To check $1'$, we observe that $[D] = \theta^1_{\lambda\lambda}$ for some λ. Hence by Lemma 4.1,

$$\begin{aligned}\overline{[D]\pi_r(b)} &= \overline{[D]}.\overline{\pi_r(b)} \\ &= [D]\pi_r(\bar{b}) \\ &= [D]\pi_r(b).\end{aligned}$$

We now verify $2'$. Property 2 for Lusztig's canonical basis implies that, by expressing $b \in B$ as a linear combination of $b', b'' \in L$, we can write

$$[D]\pi_r(b) = [D]\pi_r(b') + \sum_{b'' \neq b'} c_{b''}[D]\pi_r(b''),$$

where $[D].\pi_r(b)$ is as in the statement of the proposition and the coefficients $c_{b''}$ lie in $v^{-1}\mathcal{A}^-$. Thus by Proposition 3.10,

$$[D]\pi_r(b) \equiv [D]\pi_r(b') \mod \mathcal{L}'/v^{-1}\mathcal{L}'.$$

Suppose that $[D]\pi_r(b') = 0$. Then $[D]\pi_r(b) \in v^{-1}\mathcal{L}'$. This means that

$$[D]\pi_r(b) = \sum c_U \theta_U,$$

where $c_U \in v^{-1}\mathcal{A}^-$. (This follows from property $2'$ of the canonical basis of $S_q^+(n,r)$; the elements θ_U are an $\mathcal{A}^-$-basis of $\mathcal{L}'$.) Applying $^-$ to this equation shows that $c_U = \overline{c_U}$ for all c_U. This forces all the c_U to be zero, and $[D]\pi_r(b) = 0$.

This gives the correspondence required by $2'$, completing the proof. □

We can now give a partial result towards the positivity conjecture.

Corollary 4.3 The algebra $S_q^+(n,r)$ equipped with its canonical basis has the positivity property.

Proof First note that any canonical basis element θ_A has the property that $\theta_A = \theta_D \theta_A \theta_{D'}$ for unique diagonal matrices D and D'. Also, $[D][D'] = \delta_{D,D'}[D]$. Thus if $[D]\pi_r(b_1)$ and $[D']\pi_r(b_2)$ are both canonical basis elements (as before), then

$$[D]\pi_r(b_1)[D']\pi_r(b_2) = \begin{cases} 0 & \text{if } ([D]\pi_r(b))[D'] = 0, \\ [D]\pi_r(b_1 b_2) & \text{if } ([D]\pi_r(b))[D'] = [D]\pi_r(b). \end{cases}$$

We now invoke a deep result of Lusztig, namely the positivity property for Lusztig's canonical basis ([**14**, Theorem 14.4.13 (a)]). This shows that

$$b_1 b_2 = \sum_{b' \in B} c_{b'} b',$$

where $c_{b'} \in \mathbb{N}[v, v^{-1}]$. We deduce that

$$[D]\pi_r(b_1)[D']\pi_r(b_2) = \sum_{b' \in B} c_{b'} [D]\pi_r(b'),$$

thus establishing positivity as required. □

The final stage of the proof is to extend this partial result to the whole algebra $S_q(n,r)$. To do this, we use the following result.

Lemma 4.4 (Du) Let $\theta^d_{\lambda\mu}$, $\theta^1_{\mu,\nu}$ (where $d \in \mathcal{D}_{\lambda\mu}$) be canonical basis elements for $S_q(n,r)$. Suppose $W_\nu \subseteq W_\mu$. Then

$$\theta^d_{\lambda\mu}.\theta^1_{\mu\nu} = \theta^{d'}_{\lambda\nu}$$

where d' is such that $W_\lambda d^+ W_\nu = W_\lambda d' W^\nu$. In particular, if $W_\mu = W_\nu$ then $d' = d$.

Similarly,

$$\theta^1_{\nu\mu}\theta^{d^{-1}}_{\mu\lambda} = \theta^{d'^{-1}}_{\nu\lambda}.$$

Proof This comes from [**5**, Lemma 5.2]. $\square$

Definition For $a, b \in \{1,2,3\}$ and $n > 1$ we define

$$D(n,a,b) = \{A \in M(3n,r) : A_{i,j} \neq 0 \Rightarrow \exists x : i = x + (a-1)n,\ j = x + (b-1)n\}.$$

Example Let $a = 1$, $b = 2$. Then a typical element of $D(n,a,b)$ is a matrix

$$\begin{pmatrix} 0 & D & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where D is an $n \times n$ diagonal matrix and the zero entries represent $n \times n$ diagonal matrices with zero entries.

Definition For $i, j \in \{1,2,3\}$, we define $\sigma_{i,j} \in S_q(3n,r)$ by

$$\sigma_{i,j} = \sum_{D \in D(n,i,j)} [D].$$

Note that such a $[D]$ is of the form $\theta^1_{\lambda,\mu}$ where $W_\lambda = W_\mu$.

Lemma 4.5

(a) The injective $\mathcal{A}$-linear map $\iota : S_q(n,r) \hookrightarrow S_q(3n,r)$ given by $\iota(\theta_A) = \theta_{A'}$ where

$$A' = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is a monomorphism of $\mathcal{A}$-algebras.

(b) On $\mathrm{im}(\iota)$, $\sigma_{1,1}$ is the multiplicative identity, that is, $\sigma_{1,1}\iota(\theta_A) = \iota(\theta_A)\sigma_{1,1} = \iota(\theta_A)$

(c) For $i, j, k \in \{1, 2, 3\}$, $\sigma_{i,j}\sigma_{j,k} = \sigma_{i,k}$.

Proof Part (a) follows easily from the definition of the canonical basis. For part (b), observe that $\iota(1) = \sigma_{1,1}$, where 1 is the identity in $S_q(n, r)$. Part (c) is a corollary of Lemma 4.4. □

We are now ready to prove the main result.

Theorem 4.6 The structure constants for $S_q(n, r)$ with respect to the canonical basis lie in $\mathbb{N}[v, v^{-1}]$.

Proof Choose $\theta_A, \theta_B \in S_q(n, r)$. We may assume the product $\theta_A\theta_B$ is not zero. We express the structure constants as

$$\theta_A\theta_B = \sum_{C \in M(n,r)} f_{A,B,C}\theta_C.$$

Hence by Lemma 4.5 (a),

$$\sigma_{1,1}\iota(\theta_A)\iota(\theta_B)\sigma_{1,3} = \sum_{C \in M(n,r)} f_{A,B,C}\sigma_{1,1}\iota(\theta_C)\sigma_{1,3}.$$

Using Lemma 4.5 (b) and (c) we find that

$$(\sigma_{1,1}\iota(\theta_A)\sigma_{1,2})(\sigma_{2,1}\iota(\theta_B)\sigma_{1,3}) = \sum_{C \in M(n,r)} f_{A,B,C}(\sigma_{1,1}\iota(\theta_C)\sigma_{1,3}).$$

Note that by Lemma 4.4, all the expressions in parentheses are canonical basis elements of $S_q(3n, r)$. More importantly, the matrix indexing the basis element $\sigma_{1,1}\theta_A\sigma_{1,2}$ is of the form

$$\begin{pmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the matrix indexing the basis element $\sigma_{2,1}\theta_B\sigma_{1,3}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrices corresponding to the elements $\sigma_{1,1}\iota(\theta_C)\sigma_{1,3}$ are of the form

$$\begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These are all upper triangular canonical basis elements, so lie in $S_q^+(3n,r)$. Thus by Corollary 4.3, all the $f_{A,B,C}$ lie in $\mathbb{N}[v,v^{-1}]$. $\square$

Corollary 4.7 The structure constants for $S_q(n,r)$ with respect to the canonical basis are naturally interpretable in terms of intersection cohomology.

Proof The proof of Theorem 4.6 shows that the structure constants are the same as certain structure constants in the algebra $S_q^+(3n,r)$. The proof of Corollary 4.3 shows that these are the same as certain structure constants arising from Lusztig's canonical basis (provided they are not zero). It is known from [**14**, Theorem 14.4.13] that these latter constants have a natural interpretation in the framework of perverse sheaves and intersection cohomology. $\square$

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