FULL HEAPS AND REPRESENTATIONS OF AFFINE KAC–MOODY ALGEBRAS

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Abstract. We give a combinatorial construction, not involving a presentation, of almost all untwisted affine Kac–Moody algebras modulo their one-dimensional centres in terms of signed raising and lowering operators on a certain distributive lattice $\mathcal{B}$. The lattice $\mathcal{B}$ is constructed combinatorially as a set of ideals of a "full heap" over the Dynkin diagram, which leads to a kind of categorification of the positive roots for the Kac–Moody algebra. The lattice $\mathcal{B}$ is also a crystal in the sense of Kashiwara, and its span affords representations of the associated quantum affine algebra and affine Weyl group. There are analogues of these results for two infinite families of twisted affine Kac–Moody algebras, which we hope to treat more fully elsewhere.

By restriction, we obtain combinatorial constructions of the finite dimensional simple Lie algebras over $\mathbb{C}$, except those of types E_8, F_4 and G_2. The Chevalley basis corresponding to an arbitrary orientation of the Dynkin diagram is then represented explicitly by raising and lowering operators. We also obtain combinatorial constructions of the spin modules for Lie algebras of types B and D, which avoid Clifford algebras, and in which the action of Chevalley bases on the canonical bases of the modules may be explicitly calculated.

Contents

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Introduction

A heap is an isomorphism class of labelled posets, depending on an underlying graph Γ and satisfying certain axioms. Heaps have a wide variety of applications in algebraic combinatorics and statistical mechanics, as explained in [**21**]. The algebraic and combinatorial theory of heaps mostly concentrates on the case of finite heaps, but there is a well-developed theory of infinite heaps used in the study of parallelism in computer science, where they are known as "dependence graphs" [**5**].

In this paper, we introduce and study some remarkable infinite (but locally finite) heaps, which we call "full heaps", and which have some interesting applications to algebraic Lie theory. Let $\mathcal{B}$ denote the set of nonempty proper ideals of a full heap E (regarded as a poset) over a graph Γ. Using the poset structure, we will define a family of raising and lowering operators on the space V_E spanned by $\mathcal{B}$. If the underlying graph, Γ, of the full heap is a doubly laced Dynkin diagram associated to a symmetrizable Kac–Moody algebra (meaning that all the entries of the corresponding generalized Cartan matrix lie in the set $\{2, 0, -1, -2\}$) then we will show how the space V_E naturally carries the structure of (a) a module for the Kac–Moody algebra corresponding to Γ and (b) a module for the Weyl group corresponding to Γ. Moreover, the Chevalley generators (in the Kac–Moody case) and the Coxeter generators (in the Weyl group case) act on V_E via extremely simple raising and lowering operators applied to basis elements.

If Γ corresponds to an untwisted affine Kac–Moody algebra $\mathfrak{g}$, the representation of $\mathfrak{g}$ on V_E over $\mathbb{C}$ has a small kernel, namely the one-dimensional centre. If we restrict attention to the corresponding finite dimensional simple Lie algebra over $\mathbb{C}$, the representation will of course be faithful, but one can be much more precise: it is possible to construct the Chevalley basis arising from a given orientation of the Dynkin diagram (see [**12**, (7.8.5), (7.9.3)]) explicitly in terms of raising and lowering operators.

Raising and lowering operators are familiar in other combinatorial models of Lie theory. The most important of these include the Kashiwara operators on crystals [**13**], used in the approach of the Kyoto school, and Littelmann's path operators [**16**, **17**]. Another example of raising and lowering operators occurs in the context of the down-up algebras of Benkart and Roby [**3**].

For the case of simply laced finite dimensional simple Lie algebras over $\mathbb{C}$ (excluding E_8), a combinatorial construction for the Lie algebras by raising and lowering operators on ideals of (finite) heaps has been described by Wildberger [**22**]. Unfortunately, that paper contains no proof of its main result [**22**, Theorem 4.1], which is analogous to our Theorem 6.7, and to the best of our knowledge, no proof exists. The constructions we describe here are modified versions of Wildberger's, and have some advantages over them (see §9). Wildberger has also succeeded in dealing with the simple Lie algebra of type G_2 using raising and lowering operators [**23**], but the construction is ad hoc and significantly different from those of [**22**] or of this paper.

The constructions described above require almost no knowledge of Lie theory, apart from the definition of a Lie algebra and the notion of a Dynkin diagram (or, equivalently, a generalized Cartan matrix). In particular, the definition of a full heap is purely combinatorial. However, the proofs that the constructions work do use Lie theoretic concepts.

Our representations of Kac–Moody algebras exist whenever we have a full heap over the appropriate Dynkin diagram. This includes all untwisted affine Kac–Moody algebras except three (types $E_8^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$ in Kac's notation), and also includes two families of twisted affine Kac–Moody algebras (types $A_{2l-1}^{(2)}$ and $D_{l+1}^{(2)}$). The more complicated root systems in the twisted case make analysis more difficult, so we will concentrate almost entirely on the untwisted case in this paper for reasons of space.

Although our methods do not work for all types, they apply remarkably uniformly in the cases where they do work. The representations V_E behave like affine analogues of the minuscule representations of the corresponding simple Lie algebras; in particular, the three cases mentioned above where full heaps do not exist correspond to the three simple Lie algebras (E_8, F_4 and G_2) that have no minuscule representations (see [**19**, §2.2] for more details).

The representation of a Kac–Moody algebra on V_E has a q-analogue, namely an action of the quantum affine algebra. Regarded in this way, V_E is an integrable module and $\mathcal{B}$ is a crystal basis for V_E in the sense of Kashiwara, although it does not give an extremal weight crystal.

Although we do not emphasise this in the sequel, the labelled heaps over a fixed graph can be made into a category in which the isomorphism classes of objects are precisely the heaps. The framework of this paper can be regarded as a kind of categorification of the positive roots of (most) affine Kac–Moody algebras, in which a positive root α corresponds to a nonempty collection $\mathcal{L}_\alpha$ of labelled heaps. An element of $\mathcal{L}_\alpha$ is called a root heap of character α. Isomorphic labelled heaps have the same characters, but since the isomorphism class is not determined by the character, we do not have a categorification in the strict sense of [**1**], but rather in the weaker sense in which Khovanov homology [**15**] is a categorification of the Jones polynomial. This has the consequence that root heaps have invariants that are not invariants of the underlying positive root, and the most important for our purposes in the simply laced case is that of the parity of a root heap. This depends on an arbitrarily chosen orientation of the Dynkin diagram and, when decategorified correctly (Lemma 4.4), produces the asymmetry functions of [**12**, (7.8.4)]. We also give a decomposition of a root heap into convex sub-root heaps that corresponds to expressing a positive root as a sum of two positive roots (Corollary 5.5). In the simply laced case, this decomposition is unique, which corresponds to the fact that the structure constants for the corresponding Chevalley basis lie in the set $\{-1, 0, 1\}$. Our procedure for treating non simply laced cases is also a categorification of a well-known procedure for producing non simply laced root systems, as we discuss in §6.

The main results of the paper are as follows. The representation of the derived algebra $\mathfrak{g}'(A)$ of a symmetrizable Kac–Moody algebra is constructed in Theorem 3.1. The Chevalley bases for simple Lie algebras over $\mathbb{C}$ are constructed combinatorially in Theorem 6.7, and the corresponding result for the whole untwisted affine Kac–Moody algebra modulo its centre is given in Theorem 7.10. A q-analogue of the latter result is given in Theorem 8.3, which also explains the connection with crystal bases.

§1. Heaps over Dynkin diagrams

Let A be an n by n matrix with integer entries. Following [**12**, §1.1], we call A a *generalized Cartan matrix* if it satisfies the conditions (a) $a_{ii} = 2$ for all $1 \leq i \leq n$, (b) $a_{ij} \leq 0$ for $i \neq j$ and (c) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. In this paper, we will only consider generalized Cartan matrices with entries in the set $\{2, 0, -1, -2\}$; such matrices are sometimes called *doubly laced*. If, furthermore, A has no entries equal to -2, we will call A *simply laced*.

The Dynkin diagram $\Gamma = \Gamma(A)$ associated to a generalized Cartan matrix is a directed graph, possibly with multiple edges, and vertices indexed (for now) by the integers 1 up to n. If $i \neq j$ and $|a_{ij}| \geq |a_{ji}|$, we connect the vertices corresponding to i and j by $|a_{ij}|$ lines; this set of lines is equipped with an arrow pointing towards i if $|a_{ij}| > 1$. For example, if $a_{ij} = a_{ji} = -2$, this will result in a double edge between i and j equipped with an arrow pointing in each direction (see Figure 7 in the Appendix). There are further rules if $a_{ij}a_{ji} > 4$, but we do not need these for our purposes.

The Dynkin diagram (together with the enumeration of its vertices) and the generalized Cartan matrix determine each other, so we may write $A = A(\Gamma)$. If Γ is connected, we call A *indecomposable*.

Let Γ be a Dynkin diagram with vertex set P and no multiple edges. Let C be the relation on P such that $x \ C \ y$ if and only if x and y are distinct unadjacent vertices in Γ, and let $\mathcal{C}$ be the complementary relation.

Definition 1.1. A *labelled heap* over Γ is a triple $(E, \leq, \varepsilon)$ where $(E, \leq)$ is a locally finite partially ordered set (in other words, a poset all of whose intervals are finite) with order relation denoted by $\leq$ and where ε is a map $\varepsilon : E \longrightarrow P$ satisfying the following two axioms.

1. For every $\alpha, \beta \in E$ such that $\varepsilon(\alpha) \ \mathcal{C} \ \varepsilon(\beta)$, α and β are comparable in the order $\leq$.
2. The order relation $\leq$ is the transitive closure of the relation $\leq_{\mathcal{C}}$ such that for all $\alpha, \beta \in E$, $\alpha \ \leq_{\mathcal{C}} \ \beta$ if and only if both $\alpha \leq \beta$ and $\varepsilon(\alpha) \ \mathcal{C} \ \varepsilon(\beta)$.

We call $\varepsilon(\alpha)$ the *label* of α. In the sequel, we will sometimes appeal to the fact that the partial order is the reflexive, transitive closure of the covering relations, because of the local finiteness condition.

Definition 1.2. Let $(E, \leq, \varepsilon)$ and $(E', \leq', \varepsilon')$ be two labelled heaps over Γ. We say that E and E' are isomorphic (as labelled posets) if there is a poset isomorphism $\phi : E \longrightarrow E'$ such that $\varepsilon = \varepsilon' \circ \phi$.

A *heap* over Γ is an isomorphism class of labelled heaps. We denote the heap corresponding to the labelled heap $(E, \leq, \varepsilon)$ by $[E, \leq, \varepsilon]$.

We will sometimes abuse language and speak of the underlying set of a heap, when what is meant is the underlying set of one of its representatives.

It can be shown [**21**, §2] that the finite heaps over a graph have a well defined monoid structure, induced by an operation $\circ$ on labelled heaps which we now define.

Definition 1.3. Let $E = (E, \leq_E, \varepsilon)$ and $F = (F, \leq_F, \varepsilon')$ be two finite labelled heaps over Γ. We define the finite labelled heap $G = (G, \leq_G, \varepsilon'') = E \circ F$ over Γ as follows.

1. The underlying set G is the disjoint union of E and F.
2. The labelling map ε'' is the unique map $\varepsilon'' : G \longrightarrow P$ whose restriction to E (respectively, F) is ε (respectively, ε').
3. The order relation $\leq_G$ is the transitive closure of the relation $\mathcal{R}$ on G, where $\alpha \mathcal{R} \beta$ if and only if one of the following three conditions holds:

(i) $\alpha, \beta \in E$ and $\alpha \leq_E \beta$;

(ii) $\alpha, \beta \in F$ and $\alpha \leq_F \beta$;

(iii) $\alpha \in E,\ \beta \in F$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon'(\beta)$.

Definition 1.4. Let $(E, \leq, \varepsilon)$ be a labelled heap over Γ, and let F a subset of E. Let ε' be the restriction of ε to F. Let $\mathcal{R}$ be the relation defined on F by $\alpha \mathcal{R} \beta$ if and only if $\alpha \leq \beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$. Let $\leq'$ be the transitive closure of $\mathcal{R}$. Then $(F, \leq', \varepsilon')$ is a labelled heap over Γ. The heap $[F, \leq', \varepsilon']$ is called a *subheap* of $[E, \leq, \varepsilon]$.

If $E = (E, \leq, \varepsilon)$ is a labelled heap over Γ, then we define the *dual labelled heap*, E^* of E, to be the labelled heap $(E, \geq, \varepsilon)$. (The notion of "dual heap" is defined analogously.)

If F is convex as a subset of E (in other words, if $\alpha \leq \beta \leq \gamma$ with $\alpha, \gamma \in F$, then $\beta \in F$) then we call F a *convex subheap* of E. If, whenever $\alpha \leq \beta$ and $\beta \in F$ we have $\alpha \in F$, then we call F an *ideal* of E; dually, if, whenever $a \geq \beta$ and $\beta \in F$ we have $\alpha \in F$, then we call F a *filter* of E. If F is an ideal of E with $\emptyset \subsetneq F \subsetneq E$ such that for each vertex p of Γ we have $\emptyset \subsetneq F \cap \varepsilon^{-1}(p) \subsetneq \varepsilon^{-1}(p)$, then we call F a *proper ideal* of E.

Remark 1.5. If E and F are finite heaps over Γ, then it follows from the above two definitions that E and F are both convex subheaps of $E \circ F$, and that E is an ideal of $E \circ F$.

We will often implicitly use the fact that a subheap is determined by its set of vertices and the heap it comes from.

Definition 1.6. Let $(E, \leq, \varepsilon)$ be a locally finite labelled heap over Γ. We say that $(E, \leq, \varepsilon)$ and $[E, \leq, \varepsilon]$ are *fibred* if

(a) for each vertex p in Γ, the subheap $\varepsilon^{-1}(p)$ is unbounded above and unbounded below,

(b) for every pair p, p' of adjacent vertices in Γ and every element $\alpha \in E$ with $\varepsilon(\alpha) = p$, there exists $\beta \in E$ with $\varepsilon(\beta) = p'$ such that either α covers β or β covers α in E.

Remark 1.7.

(i) It is easily checked that these are sound definitions, because they are invariant under isomorphism of labelled heaps.

(ii) The name "fibred" alludes to the fact that these heaps can also be constructed using fibre bundles. For $x \in E$, define the set $O_x^E \subseteq E \times \Gamma$ to consist of all pairs (x, p), where $x \in E$ and there exists $y \in E$ with $\varepsilon(y) = p$ such that either x covers y or y covers x. For each vertex a of Γ, define the set $O_a^\Gamma \subseteq \Gamma \times \Gamma$ to consist of all pairs (a, b) such that a and b are adjacent in Γ. Define $\pi : E \times \Gamma \longrightarrow \Gamma \times \Gamma$ by $\pi((x, p)) = (\varepsilon(x), p)$. Let $\mathbb{Z}$ be the set of integers equipped with the discrete topology, equip $E \times \Gamma$ with the smallest topology such that the sets O_x^E are open, and equip $\Gamma \times \Gamma$ with the smallest topology such that the sets O_a^Γ are open. Then E is fibred if and only if

$$\mathbb{Z} \longrightarrow E \times \Gamma \xrightarrow{\pi} \Gamma \times \Gamma$$

is a fibre bundle.

(iii) Condition (a) provides a way to name the elements of E, which we shall need in the sequel. Choose a vertex p of Γ. Since E is locally finite, $\varepsilon^{-1}(p)$ is a chain of E isomorphic as a partially ordered set to the integers, so one can label each element of this chain as $E(p, z)$ for some $z \in \mathbb{Z}$. Adopting the convention that $E(p, x) < E(p, y)$ if $x < y$, this labelling is unique once a distinguished vertex $E(p, 0) \in \varepsilon^{-1}(p)$ has been chosen for each p.

Definition 1.8. Let E be a fibred heap over a Dynkin diagram Γ with generalized Cartan matrix A. If every open interval (α, β) of E such that $\varepsilon(\alpha) = \varepsilon(\beta) = p$ and $(\alpha, \beta) \cap \varepsilon^{-1}(p) = \emptyset$ satisfies $\sum_{\gamma \in (\alpha,\beta)} a_{p,\varepsilon(\gamma)} = -2$, we call E a *full* heap.

The above definition is reminiscent of Stembridge's definition of a minuscule heap in [**20**, §3]; however, we are following Kac's definition of generalized Cartan matrix [**12**, §4.7], which is the transpose of Stembridge's. (This distinction only applies to the matrices, and not to the corresponding heaps.) The definition says that either (a) (α, β) contains precisely two elements with labels (q_1, q_2, say) adjacent to p and such that there is no arrow from q_1 (or q_2) to p in the Dynkin diagram, or that (b) (α, β) contains precisely one element with label (q, say) adjacent to p such that there is an arrow from q to p in the Dynkin diagram.

§2. Ideals of full heaps

In §2, we develop some properties of ideals of full heaps, and use them to define raising and lowering operators.

Lemma 2.1. *Let $(E, \leq, \varepsilon)$ be a full labelled heap over Γ, let F and F' be proper ideals of E and let J be an ideal of E.*

(i) *With the labelling convention of Remark 1.7, the ideal J is proper if and only if for all vertices $p \in \Gamma$, we have*

$$J \cap \varepsilon^{-1}(p) = E_p(N) := \{E(p, t) : t < N\}$$

for some integer N depending on J and p.

(ii) *The subheaps $F \cap F'$ and $F \cup F'$ are proper ideals of E.*

(iii) *If $F \subseteq J$ and $J \backslash F$ is finite, then J is a proper ideal of E.*

(iv) *If $J \subseteq F$ and $F \backslash J$ is finite, then J is a proper ideal of E.*

(v) *If Γ is finite and connected, then J is a proper ideal if and only if $\emptyset \neq J \neq E$.*

(vi) *If Γ is finite and $F \subseteq F'$, then $F' \backslash F$ is finite.*

(vii) *If Γ is finite and connected and L is a finite convex subheap of E, then there is a proper ideal L' of E such that $L \subset L'$ and $L' \backslash L$ is a proper ideal of E.*

Proof. Part (i) follows from Remark 1.7 (iii) and the definition of proper ideal.

Part (ii) follows from (i) and the fact that, for a fixed vertex p, the chains $E_p(N)$ are closed under finite intersections and finite unions.

For part (iii), choose N so that $F \cap \varepsilon^{-1}(p) = E_p(N)$. Since $J \backslash F$ is finite,

$$(J \cap \varepsilon^{-1}(p)) \backslash (F \cap \varepsilon^{-1}(p))$$

must be finite. However, since J is an ideal, $J \cap \varepsilon^{-1}(p)$ must be downward closed (meaning that if $E(p, y) \in J \cap \varepsilon^{-1}(p)$ and $x < y$, then $E(p, x) \in J \cap \varepsilon^{-1}(p)$). This shows that $J \cap \varepsilon^{-1}(p) = E_p(N')$ for some $N' \geq N$.

Part (iv) follows by a similar argument to that used to prove part (iii), mutatis mutandis.

If J is a proper ideal, then it follows easily from (i) that $\emptyset \neq J \neq E$, as required for (v), so suppose that $\emptyset \neq J \neq E$ for an ideal J. Let $\alpha \in E \backslash J$ and let $p = \varepsilon(\alpha)$.

Suppose first that $J \cap \varepsilon^{-1}(p) \neq \emptyset$. Since $J \cap \varepsilon^{-1}(p)$ is an ideal of $\varepsilon^{-1}(p)$ and $\alpha \notin J$, we have $J \cap \varepsilon^{-1}(p) = E_p(N)$. Let q be adjacent to p in Γ. The definition of full heap ensures that there exists an element $\beta > \alpha$ with $\varepsilon(\beta) = q$, and since $\alpha \notin J$, we must have $\beta \notin J$. On the other hand, there also exists an element $\beta' < E(p, N-1)$ with $\varepsilon(\beta') = q$, and the fact that J is an ideal means that $\beta' \in J$. Combining these observations, we see that $J \cap \varepsilon^{-1}(q) = E_q(N')$ for some integer N'. Since Γ is connected, a similar condition holds at each vertex, and J is proper.

The other possibility is that $J \cap \varepsilon^{-1}(p) = \emptyset$. By reversing the argument of the above paragraph, we find that $J \cap \varepsilon^{-1}(q) = \emptyset$ for all vertices q, in other words, that $J = \emptyset$, contrary to the hypothesis of (v).

Under the assumptions of (vi), the sets $(F' \backslash F) \cap \varepsilon^{-1}(p)$ are all finite by (i), which means that $F' \backslash F$ is also finite because Γ is.

For part (vii), define $L' = \{\alpha \in E : \alpha \leq \beta \text{ for some } \beta \in L\}$. It is easily checked that L' is a nonempty ideal of E and that $L' \backslash L$ is an ideal of E. Furthermore, L' is bounded above (because L is), so $L' \neq E$, L is a proper ideal by (v) and $L' \backslash L$ is a proper ideal by (iii). □

Part (vi) of Lemma 2.1 will often be used without comment in the sequel. Part (ii) of the lemma has the following immediate corollary.

Corollary 2.2. *The set of all proper ideals of E has the structure of a distributive lattice, where $I \wedge J := I \cap J$ and $I \vee J := I \cup J$.* □

Definition 2.3. Let R^+ be the set of all functions $P \longrightarrow \mathbb{Z}^{\geq 0}$. If F is a finite labelled heap over Γ, then we define the *character*, $\chi(F)$ of F to be the element of R^+ such that $\chi(F)(p)$ is the number of elements of F with ε-value p. If $\alpha \in R^+$, we write $\mathcal{L}_\alpha(E)$ to be the set of all convex subheaps F of E with $\chi(F) = \alpha$. If F consists of a single element α with $\varepsilon(\alpha) = p$, we will write $\chi(F) = p$ for short, so that $\mathcal{L}_p(E)$ is identified with the elements of E labelled by p.

Since the function χ is an invariant of labelled heaps, we can extend the definition to apply to finite heaps of Γ.

Example 2.4. Let Γ be the Dynkin diagram of type $E_7^{(1)}$, shown in Figure 17 in the Appendix, let E be the heap shown in Figure 18, and let F be the finite convex subheap shown in the dashed box. Writing α_i for the function sending $i \in P$ to 1 and $j \in P$ to 0 if $j \neq i$, we find that

$$\chi(F) = \alpha_0 + 2\alpha_1 + 3\alpha_3 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7.$$

Lemma 2.5. *Let $[E, \leq, \varepsilon]$ be a full heap over the Dynkin diagram Γ, with generalized Cartan matrix A, let I be an ideal of E and let $\alpha \in I$ be a maximal element. Define $p = \varepsilon(\alpha)$, and suppose that $q \in \Gamma$ is adjacent to p. Then precisely one of the following occurs:*

(i) *$a_{qp} = -1$, $I \backslash \{\alpha\}$ is an ideal of E and there exists a maximal element $\beta \in I \backslash \{\alpha\}$ with $\varepsilon(\beta) = q$,*

(ii) *$a_{qp} = -1$, there exists a minimal element $\beta \in E \backslash I$ such that $\varepsilon(\beta) = q$ and $I \cup \{\beta\}$ is an ideal of E, or*

(iii) *$a_{qp} = -2$, there exists a maximal element $\beta \in I \backslash \{\alpha\}$ and a minimal element $\beta' \in E \backslash I$ such that $\varepsilon(\beta) = \varepsilon(\beta') = q$, and both $I \backslash \{\alpha\}$ and $I \cup \{\beta'\}$ are ideals of E.*

Proof. By part (b) of the definition of a fibred heap, there exists $\beta \in E$ with $\varepsilon(\beta) = q$ such that either α covers β or β covers α. Until further notice, let us assume that $a_{qp} = -1$.

Suppose first that $\beta < \alpha$. Since α is maximal in I, it follows that $I \backslash \{\alpha\}$ is an ideal of E. If β is maximal in $I \backslash \{\alpha\}$, then we are in the situation of (i) above, so suppose this is not the case. By part (a) of the definition of a fibred heap, there exists $\gamma \in E$ with $\varepsilon(\gamma) = q$ and $\gamma > \beta$. Since $\{\alpha, \beta, \gamma\}$ is a chain in E and $\beta < \alpha$ is a covering relation, we have $\gamma > \alpha$ and $\gamma \notin I$. Because $\varepsilon^{-1}(q) \cup \{\alpha\}$ is a chain in E, we may assume that the interval (β, γ) of E contains no elements of $\varepsilon^{-1}(q)$. By the definition of full, (β, γ) contains two elements, γ_1 and γ_2, with labels adjacent to q, and one of these elements, γ_1 say, is α. The hypothesis that β is not maximal in $I \backslash \{\alpha\}$ implies that γ_2 is also in I. We claim now that $I \cup \{\gamma\}$ is an ideal of E; to show this, it is enough to show that if $\gamma > \gamma'$ is a covering relation, then $\gamma' \in I$. The latter holds because any such γ' is comparable to β, which has the same label as γ', and $I \cup \{\gamma\}$ contains all elements less than β together with the closed interval $[\beta, \gamma]$. This satisfies the conditions of (ii).

Suppose now that $\beta > \alpha$. It is clear that β is minimal in $E \backslash I$, so that if $I \cup \{\beta\}$ is an ideal of E, the conditions of (ii) will hold. Suppose that this is not the case. By part (a) of the definition of a fibred heap, there exists $\gamma \in E$ with $\varepsilon(\gamma) = q$ and $\gamma < \beta$. As in the previous paragraph, this means that $\gamma < \alpha$, from which we see that $\gamma \in I$. By the definition of full, (γ, β) contains two elements, γ_1 and γ_2, with labels adjacent to q, and one of these elements, γ_1 say, is α. If γ_2 also lies in I, then the conditions of (ii) will be satisfied as in the previous paragraph. If γ_2 does not lie in I, then $I \cup \{\beta\}$ is not an ideal, but $I \backslash \{\alpha\}$ is an ideal with maximal element γ, and $\varepsilon(\gamma) = q$. This satisfies the conditions of (i).

From now on, assume that $a_{qp} = -2$. Suppose there exists $\beta \in E$ with $\varepsilon(\beta) = q$ such that α covers β. By part (a) of the definition of a fibred heap, there exists $\gamma \in E$ with $\varepsilon(\gamma) = q$ and $\gamma > \beta$. Following the same reasoning as earlier, we have $\gamma > \alpha$ and $\gamma \notin I$. The other possibility is that there exists $\beta \in E$ with $\varepsilon(\beta) = q$ such that β covers α. In this case, part (a) of the definition of a fibred heap shows that there exists $\gamma \in E$ with $\varepsilon(\gamma) = q$ and $\gamma < \beta$. As before, this means that $\gamma < \alpha$, from which we see that $\gamma \in I$. In either case, there exists a chain $\beta_1 < \alpha < \beta_2$ in E with $\varepsilon(\beta_1) = \varepsilon(\beta_2) = q$, such that the open interval (β_1, β_2) contains no elements labelled q. By the definition of full, α is the only element in (β_1, β_2) with a label adjacent to q, and thus the only element in (β_1, β_2), meaning that $\{\beta_1, \alpha, \beta_2\}$ is a convex chain. The assertions of case (iii) now follow by adapting the argument for the case $a_{qp} = -1$. $\square$

The following definition generalizes ideas in [Wi, §4].

Definition 2.6. Let E be a full heap over a graph Γ, let k be a field (always of characteristic not equal to 2), and define V_E to be the k-span of the distributive lattice

$$\mathcal{B} = \{v_I : I \text{ is a proper ideal of } E\}.$$

For any such ideal and any finite convex subheap $L \leq E$, we write $L \succ I$ to mean that both $I \cup L$ is an ideal and $I \cap L = \emptyset$, and we write $L \prec I$ to mean that both $L \leq I$ and $I \backslash L$ is an ideal. We define linear operators X_L, Y_L and H_L on V_E as follows:

$$X_L(v_I) = \begin{cases} v_{I \cup L} & \text{if } L \succ I, \\ 0 & \text{otherwise,} \end{cases}$$
$$Y_L(v_I) = \begin{cases} v_{I \backslash L} & \text{if } L \prec I, \\ 0 & \text{otherwise,} \end{cases}$$
$$H_L(v_I) = \begin{cases} v_I & \text{if } L \prec I \text{ and } L \not\succ I, \\ -v_I & \text{if } L \succ I \text{ and } L \not\prec I, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that these operators are defined by parts (iii) and (iv) of Lemma 2.1; they are nonzero by part (vii) of Lemma 2.1.) If p is a vertex of Γ, we write X_p for the linear operator on V_E given by $\sum_{L \in \mathcal{L}_p(E)} X_L$ (with notation as in Definition 2.3), and we define Y_p and H_p similarly. Note that although these sums are infinite, it follows from the definitions of fibred and full heaps that at most one of the terms in each case may act in a nonzero way on any given v_I. In this situation, we also write $p \succ I$ to mean that $L \succ I$ for some (necessarily unique) $L \in \mathcal{L}_p(E)$, and analogously we write $p \prec I$ with the obvious meaning. Note that it is not possible for both $p \prec I$ and $p \succ I$, because I cannot contain a convex chain $\alpha < \beta$ with $\varepsilon(\alpha) = \varepsilon(\beta) = p$.

Lemma 2.7. *Maintain the above notation and suppose that p and q are vertices of Γ (allowing the possibility $p = q$). We have the following relations in the associative*

k-algebra generated by the operators X_L, Y_L and H_L, where δ is the Kronecker delta:

$$H_pH_q = H_qH_p, \tag{1}$$

$$H_pX_q - X_qH_p = a_{pq}X_q, \tag{2}$$

$$H_pY_q - Y_qH_p = -a_{pq}X_q, \tag{3}$$

$$X_pY_q - Y_qX_p = \delta_{pq}H_q, \tag{4}$$

$$X_pX_q = X_qX_p, \text{ if } a_{pq} = 0, \tag{5}$$

$$Y_pY_q = Y_qY_p, \text{ if } a_{pq} = 0, \tag{6}$$

$$X_pX_p = Y_pY_p = 0, \tag{7}$$

$$X_pX_qX_p = Y_pY_qY_p = 0 \text{ if } a_{pq} = -1. \tag{8}$$

Proof. Relation (1) holds because of the way the operators H_p act as scalars on each basis vector v_I.

Consider the algebra element $H_pX_p - X_pH_p$. This element, and X_p, will each act as zero on v_I unless $p \succ I$. If, on the other hand, $p \succ I$, let $L \succ I$ be such that $L = \{\alpha\}$ with $\varepsilon(\alpha) = p$. We then have $H_pX_pv_I = v_{I\setminus L} = -X_pH_pv_I$, and relation (2) follows.

Suppose that $a_{pq} = 0$, so that p and q are not adjacent, and consider $H_pX_q - X_qH_p$. Unless $q \succ I$, this element will act as zero on v_I, so we may reduce consideration to this case. Let $L \succ I$ be such that $L = \{\alpha\}$ with $\varepsilon(\alpha) = q$. A simple case by case check shows that $p \prec I \cup L$ (respectively, $p \succ I \cup L$) if and only $p \prec I$ (respectively, $p \succ I$), and relation (2) follows.

Now suppose that $a_{pq} = -1$, and consider $H_pX_q - X_qH_p$. As before, relation (2) follows trivially unless $q \succ I$, so we reduce to this case. Let $L \succ I$ be such that $L = \{\alpha\}$ with $\varepsilon(\alpha) = q$, so that $X_qv_I = v_{I\cup L}$. By Lemma 2.5, we have either $p \succ I \cup L$ or $p \prec I$, but not both. (Note that if $p \succ I \cup L$, then $p \not\succ I$, and if $p \prec I$, then $p \not\prec I \cup L$.) If $p \succ I \cup L$, we have $H_pX_qv_I = -v_{I\cup L}$ and $X_qH_pv_I = 0$. On the other hand if $p \prec I$, we have $H_pX_qv_I = 0$ and $X_qH_pv_I = v_{I\cup L}$. Relation (2) now follows.

Finally, suppose that $a_{pq} = -2$, and consider $H_pX_q - X_qH_p$. Again, let $L \succ I$ be such that $L = \{\alpha\}$ with $\varepsilon(\alpha) = q$, so that $X_qv_I = v_{I\cup L}$. By Lemma 2.5, we have both $p \succ I \cup L$ and $p \prec I$, meaning that $H_pX_qv_I = -v_{I\cup L}$ and $X_qH_pv_I = -v_{I\cup L}$. This completes the proof of relation (2).

The verification of relation (3) follows a similar line of argument to that used to prove relation (2), mutatis mutandis.

It follows from the definition of H_p that $X_pY_p - Y_pX_p = H_p$, thus establishing the case $p = q$ of relation (4). If p and q are adjacent, then a case by case check shows that the operators X_pY_q and Y_qX_p are individually zero. On the other hand, if p and q are not adjacent, an argument like that used on the $a_{pq} = 0$ case of relation (2) shows that X_pY_q and Y_qX_p commute, thus finishing the proof of relation (4). The same reasoning also establishes the commutation relations (5) and (6).

Relation (7) holds because no ideal I can contain a convex chain $\alpha < \beta$ with $\varepsilon(\alpha) = \varepsilon(\beta) = p$, and relation (8) holds because no ideal I can contain a convex chain $\alpha < \beta < \gamma$ with $\varepsilon(\alpha) = \varepsilon(\gamma) = p$ and $\varepsilon(\beta) = q$ if $a_{pq} = -1$. (Both of these conditions come from the definition of full.) $\square$

§3. Lie algebras and root systems

A *Lie algebra* over a field k is a k-vector space $\mathfrak{g}$ endowed with a bilinear (usually nonassociative) multiplication $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$. The image of the pair (x, y) under this map is denoted by $[x, y]$, and the following axioms hold for all elements $x, y, z \in \mathfrak{g}$:

$$[x, x] = 0;$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

The first condition above is known as *antisymmetry* and the second is known as the *Jacobi identity*. Any associative algebra A over k, such as the algebra of Lemma 2.7, may be made into a Lie algebra using the bracket $[a, b] := ab - ba$.

The significance of Lemma 2.7 is that it gives V_E the structure of a module for a certain Lie algebra, namely (in the case $k = \mathbb{C}$) the derived algebra $\mathfrak{g}'(A)$ of a symmetrizable Kac–Moody algebra (see [**12**, §0.3]). The main purpose of this paper is to understand this module.

Theorem 3.1. *Let E be a full heap over a Dynkin diagram Γ with vertices P and generalized Cartan matrix A, let k be a field of characteristic different from 2, and let V_E be the k-vector space of Definition 2.6. Let $\mathfrak{g}$ be the Lie algebra with generators $\{e_i, f_i, h_i : i \in P\}$ and the usual defining relations (see* [**4**, §9.4]*). Then V_E becomes a left $\mathfrak{g}$-module, where e_i (respectively, f_i, h_i) acts as X_i (respectively, Y_i, H_i).*

Proof. This is a consequence of Lemma 2.7, recalling that we have

$$[a, b].v := a(b(v)) - b(a(v)).$$

Note that the relation $[e_i, [e_i, e_j]] = 0$ expands to

$$e_i \circ e_i \circ e_j - e_i \circ e_j \circ e_i + e_j \circ e_i \circ e_i = 0,$$

which holds by relations (7) and (8) of Lemma 2.7, and the relation

$$[e_i, [e_i, [e_i, e_j]]] = 0$$

expands to

$$e_i \circ e_i \circ e_i \circ e_j - 2e_i \circ e_i \circ e_j \circ e_i + 2e_i \circ e_j \circ e_i \circ e_i - e_i \circ e_j \circ e_i \circ e_i = 0,$$

which holds by relation (7) of Lemma 2.7. Similar comments hold for the relations involving f_i and f_j. $\square$

To understand Lie algebras such as those in Theorem 3.1, one needs the concept of a root system, and we will show that the combinatorics of full heaps is intimately connected to that of root systems for affine Kac–Moody algebras.

We introduce the following notation in order to state later results easily.

Definition 3.2. Let L be a finite convex heap of a full heap E over a graph Γ, and let p be a vertex of Γ. We write $p{\rightarrow}L$ (respectively, $L{\leftarrow}p$) to mean that L has a minimal (respectively, maximal) vertex with label p. We write $p{\leftarrow}L$ (respectively, $L{\rightarrow}p$) to mean that there is a (necessarily unique) vertex α of $E \backslash L$ labelled p such that $L \cup \{\alpha\}$ is convex and $p{\rightarrow}L \cup \{\alpha\}$ (respectively, $p{\leftarrow}L \cup \{\alpha\}$). Define the integers $b^{\pm}(L, p)$ by the conditions

$$b^+(L,p) = \begin{cases} 1 & \text{if } L{\leftarrow}p, \\ -1 & \text{if } L{\rightarrow}p, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b^-(L,p) = \begin{cases} 1 & \text{if } p{\rightarrow}L, \\ -1 & \text{if } p{\leftarrow}L, \\ 0 & \text{otherwise.} \end{cases}$$

The integers $b^{\pm}(L, p)$ are well-defined by the definition of full heap.

The following lemma will be used repeatedly in the sequel, sometimes without explicit comment.

Lemma 3.3. *Let E be a full heap over a Dynkin diagram Γ, let k be a field, let L be a finite convex subheap of E and let p be a vertex of Γ. Then we have $[H_p, X_L] = cX_L$ and $[H_p, Y_L] = -cX_L$ for some $c \in \{-2, -1, 0, 1, 2\}$. More precisely, we have*

$$\begin{aligned} [H_p, X_L] &= b^+(L,p) + b^-(L,p), \\ [H_p, Y_L] &= -b^+(L,p) - b^-(L,p). \end{aligned}$$

Proof. This follows from the definition of H_p and a case by case check, similar to (but easier than) the proof of Lemma 2.7. $\square$

We define the Weyl group, $W(\Gamma)$, associated to Γ to be the group with generators

$\{s_i \in I\}$ indexed by the vertices of Γ and defining relations

$$s_i^2 = 1 \text{ for all } i \in I,$$

$$s_i s_j = s_j s_i \text{ if } a_{ij} = 0,$$

$$s_i s_j s_i = s_j s_i s_j \text{ if } a_{ij} < 0 \text{ and } a_{ij} a_{ji} = 1,$$

$$s_i s_j s_i s_j = s_j s_i s_j s_i \text{ if } a_{ij} < 0 \text{ and } a_{ij} a_{ji} = 2.$$

Note that no relation is added in the case where $a_{ij} < 0$ and $a_{ij} a_{ji} = 4$.

Example 3.4. Define two generalized Cartan matrices

$$A_1 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Then the Weyl group corresponding to A_1 is

$$\langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^4 = 1 \rangle,$$

isomorphic to the dihedral group of order 8, and the Weyl group corresponding to A_2 is the infinite group

$$\langle s_1, s_2 : s_1^2 = s_2^2 = 1 \rangle.$$

Let $\Pi = \{\alpha_i : i \in I\}$ and let $\Pi^\vee = \{\alpha_i^\vee : i \in I\}$. We have a $\mathbb{Z}$-bilinear pairing $\mathbb{Z}\Pi \times \mathbb{Z}\Pi^\vee \longrightarrow \mathbb{Z}$ defined by

$$\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij},$$

where (a_{ij}) is the generalized Cartan matrix. If k is a field, we extend this to a k-bilinear pairing by extension of scalars. If $v = \sum_{i \in I} \lambda_i \alpha_i$, we write $v \geq 0$ to mean that $\lambda_i \geq 0$ for all i, and we write $v > 0$ to mean that $\lambda_i > 0$ for all i. We view $V = k\Pi$ as the underlying space of a reflection representation of W, determined by the equalities $s_i(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i$ for all $i \in I$.

Indecomposable generalized Cartan matrices come in three mutually exclusive types.

Theorem 3.5 [**12**, Theorem 4.3]**.** *Let A be an indecomposable generalized Cartan matrix. Then A satisfies one and only one of the following three possibilities:*

(i) $\det A \neq 0$*; there exists $u > 0$ with $Au > 0$; and $Av \geq 0$ implies $v > 0$ or $v = 0$;*

(ii) *A has corank 1; there exists $u > 0$ with $Au = 0$; and $Av \geq 0$ implies $Av = 0$;*

(iii) *there exists $u > 0$ with $Au < 0$; and the conditions $Av \geq 0$ and $v \geq 0$ together imply $v = 0$.*

The matrix A is said to be of finite (respectively, affine, indefinite) type if it satisfies condition (i) (respectively, (ii), (iii)) above. □

In this paper, we are only concerned with the finite and affine cases above.

Following [**12**, §5], we define a *real root* to be a vector of the form $w(\alpha_i)$, where $w \in W$ and α_i is a basis vector. If A is of finite type, all roots are real. If A is of affine type, there is a unique vector $\delta = \sum a_i \alpha_i$ such that $A\delta = 0$ and the a_i are relatively prime positive integers. Although the notion of *imaginary root* can be defined in general, in the affine type case the imaginary roots are easily characterized as precisely those vectors of the form $n\delta$ where n is a nonzero integer.

A *root* is by definition a real or imaginary root. We denote the set of roots by Δ, as in [**12**]. We say a root α is positive (respectively, negative) if $\alpha > 0$ (respectively, $\alpha < 0$). If α is a root, then so is $-\alpha$, and every root is either positive or negative. We will identify the positive (real and imaginary) roots with elements of R^+ as in Definition 2.3 so that $\sum a_i \alpha_i$ corresponds to the function sending each i to a_i. The *height*, $\mathrm{ht}(\alpha)$ of the root $\alpha = \sum a_i \alpha_i$ is by definition the integer $\sum a_i$.

Lemma 3.6. *Let A_0 be a simply laced generalized Cartan matrix of finite type, and let $\alpha = \sum a_i \alpha_i$ and β be two positive roots associated to A_0. Define $\alpha^\vee = \sum a_i \alpha_i^\vee$, and write $\langle \beta, \alpha^\vee \rangle$ for $\sum a_i \langle \beta, \alpha_i^\vee \rangle$. Then precisely one of the following situations occurs:*

(i) $\langle \beta, \alpha^\vee \rangle = 2$ *and $\alpha = \beta$;*

(ii) $\langle \beta, \alpha^\vee \rangle = 1$*, $\alpha - \beta$ is a root and $\alpha + \beta$ is not a root;*

(iii) $\langle \beta, \alpha^\vee \rangle = 0$ *and neither of $\alpha \pm \beta$ is a root;*

(iv) $\langle \beta, \alpha^\vee \rangle = -1$, *$\alpha + \beta$ is a root and $\alpha - \beta$ is not a root;*

(v) $\langle \beta, \alpha^\vee \rangle = -2$ *and $\alpha = -\beta$.*

Proof. This is well-known, and the proof follows from the argument given in [**11**, §9.4]. $\square$

§4. Parity of heaps in the simply laced case

Let A be a generalized Cartan matrix, let $\mathfrak{g}$ be the associated Lie algebra, and let Γ be the corresponding Dynkin diagram. In §4, we assume that A is simply laced; in other words, that A has entries in the set $\{2, 0, -1\}$.

Let us now fix an orientation of Γ.

Definition 4.1. Following [**12**, (7.8.4)], we define a function

$$\operatorname{sgn} : P \times P \longrightarrow \{\pm 1\}$$

(depending on the chosen orientation of Γ) by the conditions

$$\operatorname{sgn}(p, p') = \begin{cases} -1 & \text{if } p = p' \text{ or there is an arrow from } p \text{ to } p', \\ 1 & \text{otherwise.} \end{cases}$$

We may extend the above definition to a function $\operatorname{sgn} : R^+ \times R^+ \longrightarrow \mathbb{Z}$ via

$$\operatorname{sgn}(f, g) = \sum_{p \in P} \sum_{q \in P} f(p) g(q) \operatorname{sgn}(p, q);$$

similarly, we may extend the definition to a function on roots by

$$\operatorname{sgn}\left(\sum a_i \alpha_i, \sum b_j \alpha_j\right) = \sum_i \sum_j a_i b_j \operatorname{sgn}(i, j).$$

Lemma 4.2. *Assume additionally that A is of finite type, and that α, β are positive roots such that $\alpha + \beta$ is a root. Then $\operatorname{sgn}(\alpha, \beta) = -\operatorname{sgn}(\beta, \alpha)$.*

Proof. This follows by combining (7.8.7) and (7.8.8) in [**12**]. $\square$

We use the above to define a parity function on finite heaps as follows.

Definition 4.3. If F is a finite labelled heap over Γ, we define

$$\operatorname{sgn}(F) = \prod_{\substack{\alpha, \beta \in F \\ \alpha > \beta}} \operatorname{sgn}(\varepsilon(a), \varepsilon(\beta)).$$

We extend the notion of parity to finite heaps over Γ, in the obvious way. (See Example 6.4 below for a sample calculation.)

For our purposes, the key purpose of the sgn-function is the following

Lemma 4.4. *Let $\gamma, \gamma' \in R^+$, and let $F_1 \in \mathcal{L}_\gamma$ and $F_2 \in \mathcal{L}_{\gamma'}$ be finite labelled heaps over Γ. Then we have*

$$\operatorname{sgn}(F_1 \circ F_2) = \operatorname{sgn}(F_1) \operatorname{sgn}(F_2) \operatorname{sgn}(\gamma', \gamma).$$

Proof. In the computation of the left hand side using Definition 4.3, three types of terms appear in the product: (a) those where α and β both lie in F_1, (b) those where α and β both lie in F_2 and (c) those where α lies in F_1 and β lies in F_2. The factorization in the statement expresses this decomposition. □

Definition 4.5. As in [**22**], we define for each $\alpha \in R^+$ operators X_α and Y_α on V_E by the formulae

$$X_\alpha = \sum_{L \in \mathcal{L}_\alpha(E)} \operatorname{sgn}(L) X_L$$

and

$$Y_\alpha = \sum_{L \in \mathcal{L}_\alpha(E)} \operatorname{sgn}(L) Y_L.$$

Although the sums in the above definition may be infinite, note that at most one summand can act as zero on any given v_I.

Lemma 4.6. *Maintain the notation of Definition 2.6, and let $L \in \mathcal{L}_\alpha$ and $L' \in \mathcal{L}_\beta$ for some $\alpha, \beta \in R^+$. If v_I is a basis element such that $X_L \circ X_{L'}(v_I) \neq 0$, then we have*

$$X_L \circ X_{L'}(v_I) = \operatorname{sgn}(\alpha, \beta) X_{L \cup L'}(v_I);$$

similarly, if v_I *is such that* $Y_L \circ Y_{L'}(v_I) \neq 0$ *then*

$$Y_L \circ Y_{L'}(v_I) = \operatorname{sgn}(\beta, \alpha) Y_{L \cup L'}(v_I).$$

Proof. This is a consequence of the definitions and Lemma 4.4. The first identity corresponds to the case $L' \circ L = L \cup L'$, and the second to the case $L \circ L' = L \cup L'$. □

§5. Representability of roots in the simply laced finite type case

In §5, we concentrate on the case of the simply laced, finite type case. However, we first require some general results (Definition 5.1 and lemmas 5.2 and 5.3), which will also be needed in later sections.

Definition 5.1. If α is a positive root associated to a Kac–Moody algebra $\mathfrak{g}$, then (identifying α with an element of R^+ in the usual way) we call elements of $\mathcal{L}_\alpha$ *root heaps*. If $\mathcal{L}_\alpha$ is nonempty, we say that the root α is *representable* in the heap E.

Lemma 5.2. *Let* $\mathfrak{g}$ *be a Kac–Moody algebra and let* α *be a real non-simple, positive root associated to* $\mathfrak{g}$*. Then there exists a simple root* α_i *such that* $\langle \alpha, \alpha_i^\vee \rangle > 0$ *and the root* $s_i(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$ *is positive.*

Proof. This is [**12**, Proposition 5.1 (e)]. □

Lemma 5.3. *Let* $\mathfrak{g}$ *be a Kac–Moody algebra with associated Dynkin diagram* Γ*, let* $\alpha = \sum a_i \alpha_i \in R^+$ *be such that* $\alpha > 0$*, let* $a = \langle \alpha, \alpha_i^\vee \rangle$*, let* E *be a full heap over* Γ*, and let* $F \leq E$ *with* $F \in \mathcal{L}_\alpha$*.*

(i) *If* $a = 2$ *then* F *has both a maximal vertex* β *and a minimal vertex* β' *with* $\varepsilon(\beta) = \varepsilon(\beta') = i$*.*

(ii) *If* $a = 1$ *then either* F *has a maximal vertex* β *with* $\varepsilon(\beta) = i$*, or minimal vertex* β' *with* $\varepsilon(\beta') = i$*, but not both.*

(iii) *If* $a = 0$ *and* F *has a maximal element labelled* i*, then there exists* $\beta \in E \backslash F$ *with* $\varepsilon(\beta) = i$ *such that* $F' = F \cup \{\beta\}$ *is convex and* β *is minimal in* F'*.*

(iv) *If $a = 0$ and F has a minimal element labelled i, then there exists $\beta \in E \backslash F$ with $\varepsilon(\beta) = i$ such that $F' = F \cup \{\beta\}$ is convex and β is maximal in F'.*

Proof. Let us write $\alpha = \sum a_k \alpha_k$ as a sum of simple roots. Since $a > 0$, we must have $a_i > 0$ for some i; in other words, F contains at least one element labelled i. Let ζ_0 and ζ_1 denote the least and greatest elements of $F \cap \varepsilon^{-1}(i)$, respectively.

In order to calculate $\langle \alpha, \alpha_i^\vee \rangle$, the only relevant summands in the expression for α are those corresponding to α_i itself and to the simple roots adjacent to α_i. Define F' to be the set of all $\gamma \in F$ with $\varepsilon(\gamma)$ adjacent to i. The definition of full heap shows that there are three possibilities for elements $\gamma \in F'$: (a) $\gamma < \zeta_0$, (b) $\gamma > \zeta_1$, or (c) γ lies between two elements ζ, ζ' of F with $\varepsilon(\zeta) = \varepsilon(\zeta') = i$ and $(\zeta, \zeta') \cap \varepsilon^{-1}(i) = \emptyset$.

Let us first consider case (c). If $a_{ij} = -2$, such an open interval (ζ, ζ') contains a unique element γ with label adjacent to i. The other possibility is that $a_{ij} = -1$, in which case (ζ, ζ') contains precisely two elements, γ and γ', with labels adjacent to i. Let α' be the element of R^+ given by the character $\chi([\zeta_0, \zeta_1])$ of the closed interval $[\zeta_0, \zeta_1]$. The above case analysis in terms of a_{ij} shows that $\langle \alpha', \alpha_i^\vee \rangle = \langle \alpha_i, \alpha_i^\vee \rangle = 2$, and this identity also holds in the case that $[\zeta_0, \zeta_1]$ consists of a single element.

The contributions to $\langle \alpha, \alpha_i^\vee \rangle$ that do not come from $\langle \alpha', \alpha_i^\vee \rangle$ must therefore come from the elements γ in cases (a) and (b) above, which means in particular that $\langle \alpha, \alpha_i^\vee \rangle \leq \langle \alpha', \alpha_i^\vee \rangle$.

If $a = 2$, then there cannot be any such elements, or the latter inequality would be strict, contrary to hypothesis. This means that ζ_0 (respectively, ζ_1) is minimal (respectively, maximal) in F, which establishes assertion (i).

If $a = 1$, then there must be precisely one such element γ (and, in fact, $a_{ij} = -1$ must also hold). This means that either ζ_0 is minimal in F, or ζ_1 is maximal in F, but not both, which establishes assertion (ii).

Suppose $a = 0$ and that there is a maximal element labelled i. There are then no elements γ corresponding to case (a). Furthermore, either there is precisely one element γ arising from case (b) and $a_{ij} = -2$, or there are precisely two elements

γ arising from case (b) and $a_{ij} = -1$. If we write $\zeta_0 = E(i,t)$, then setting $\beta = E(i, t-1)$ will then satisfy the hypothesis of (iii) by the definition of full heap.

Part (iv) is proved by a symmetrical argument. $\square$

We now let A be a simply laced generalized Cartan matrix of (necessarily untwisted) affine type, corresponding to the finite type matrix A_0. The next result, whose method of proof is familiar from Kashiwara's celebrated Grand Loop [**13**, §4], establishes the basic properties of root heaps and the operators of Definition 4.5.

Proposition 5.4. *Let E be a full heap over the Dynkin diagram Γ of A, let A_0 be the corresponding finite type generalized Cartan matrix with Kac–Moody algebra $\mathfrak{g}_0$, and let $\alpha = \sum \lambda_i \alpha_i$ be a positive real root associated to $\mathfrak{g}_0$.*

(i) *The root α is representable in E.*

(ii) *The operator X_α is nonzero, lies in the Lie algebra generated by the X_p, and (in the case $k = \mathbb{C}$) is equal to the element E_α in the notation of* [**12**, (7.8.5)].

(iii) *If p is any vertex of Γ, then $[H_p, X_\alpha] = \langle \alpha, \alpha_p^\vee \rangle X_\alpha$.*

(iv) *The operator Y_α is nonzero, lies in the Lie algebra generated by the Y_p, and (in the case $k = \mathbb{C}$) is equal to the element $-E_{-\alpha}$ in the notation of* [**12**, (7.8.5)].

(v) *If p is any vertex of Γ, then $[H_p, Y_\alpha] = -\langle \alpha, \alpha_p^\vee \rangle Y_\alpha$.*

(vi) *For any proper ideal I of E, there do not exist root heaps $L, L' \in \mathcal{L}_\alpha$ such that both $I \cup L$ and $I \backslash L'$ are ideals.*

(vii) *We have $H_\alpha = \sum \lambda_i H_{\alpha_i} = \alpha^\vee$, where $\alpha^\vee$ is as defined in* [**12**, §5.1].

Proof. We will prove the statements simultaneously by induction on $\mathrm{ht}(\alpha)$. The proofs of (iv) and (v) are very similar to those of (ii) and (iii), respectively, so we do not include them.

The base case is $\mathrm{ht}(\alpha) = 1$, in other words, α is simple. Parts (i) and (ii) follow from part (a) of the definition of a fibred heap and the definitions of [**12**, §7.8], part (iii) is immediate from Theorem 3.1, part (vi) follows from the definition of a full heap and part (vii) is trivial (again using the definitions of [**12**, §7.8]).

For the inductive step, we use Lemma 5.2 to find a simple root α_i such that $a = \langle \alpha, \alpha_i^\vee \rangle > 0$ and the root $\alpha' = s_i(\alpha) = \alpha - a\alpha_i$ is positive. By the inductive hypothesis, we have

$$[H_i, X_{\alpha'}] = \langle \alpha', \alpha_i^\vee \rangle = \langle \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i, \alpha_i^\vee \rangle = \langle \alpha, \alpha_i^\vee \rangle (1 - \langle \alpha_i, \alpha_i^\vee \rangle) = - \langle \alpha, \alpha_i^\vee \rangle .$$

Since α' is representable by the inductive hypothesis, Lemma 3.3 shows that we have $\langle \alpha, \alpha_i^\vee \rangle \in \{1, 2\}$.

This gives three possibilities for a root heap $L \in \mathcal{L}_{\alpha'}$. Case (a) is that $a = 2$, $i \leftarrow L$ and $L \rightarrow i$, but in fact this cannot occur, because it implies (by the inductive hypothesis) that $\langle \alpha', \alpha_i^\vee \rangle = -2$, which contradicts Lemma 3.6. Case (b) is that $a = 1$, and $i \leftarrow L$ and but we do not have either $L \rightarrow i$, $L \leftarrow i$ or $i \rightarrow L$. Case (c) is that $a = 1$, and $L \rightarrow i$ but we do not have either $i \leftarrow L$, $i \rightarrow L$ or $L \leftarrow i$.

We are now in case (b) or (c), so that $a = 1$. In this situation, α is representable because we can add a new maximal or minimal vertex labelled i to L to form a root heap in $\mathcal{L}_\alpha$. By Lemma 2.1 (vii), this means that X_α is nonzero, and by Lemma 5.3, any root heap in $\mathcal{L}_\alpha$ has either a maximal or minimal vertex (but not both) labelled i. We claim in this case that $X_\alpha = \pm [X_{\alpha_i}, X_{\alpha'}]$. It is enough to check that each side of the equation acts in the same way on a basis vector, v_I. The right hand side of the equation is equal to

$$X_{\alpha_i} \circ X_{\alpha'} - X_{\alpha'} \circ X_{\alpha_i}.$$

It now follows that unless we have $L' \succ I$ for some $L' \in \mathcal{L}_\alpha$, both sides of the equation will act as zero, so let us assume that this condition is satisfied. By Lemma 5.3 (ii), every element L' of $\mathcal{L}_\alpha$ is uniquely of the form $\{\beta\} \circ L$ or of the form $L \circ \{\beta\}$ (but not both) for some β and $L \in \mathcal{L}_{\alpha'}$, and we have shown that any such $L \in \mathcal{L}_{\alpha'}$ can be extended to an element of $\mathcal{L}_\alpha$ in this way.

In case (b), $X_{\alpha_i} \circ X_{\alpha'}$ acts as zero on v_I, so by Lemma 4.6 we have

$$[X_{\alpha_i}, X_{\alpha'}].v_I = -X_{\alpha'} \circ X_{\alpha_i} v_I = - \operatorname{sgn}(\alpha', \alpha_i) X_\alpha . v_I.$$

In case (c), $X_{\alpha'} \circ X_{\alpha_i}$ acts as zero on v_I, so by Lemma 4.6 we have

$$[X_{\alpha_i}, X_{\alpha'}].v_I = X_{\alpha_i} \circ X_{\alpha'} v_I = \operatorname{sgn}(\alpha_i, \alpha') X_\alpha.v_I.$$

Lemma 4.2 now shows that $[X_{\alpha_i}, X_{\alpha'}] = \operatorname{sgn}(\alpha_i, \alpha') X_\alpha$. Since, by [**12**, (7.8.5)], we have $[E_{\alpha_i}, E_{\alpha'}] = \operatorname{sgn}(\alpha_i, \alpha') E_\alpha$ (a formula also valid for negative roots), we have $X_\alpha = E_\alpha$ by the inductive hypothesis, completing the proof of (i) and (ii).

We now observe, using the Jacobi identity and the inductive hypothesis, that

$$\begin{aligned}
[H_p, X_\alpha] &= \operatorname{sgn}(\alpha', \alpha_i)[H_p, [X_{\alpha_i}, X_{\alpha'}]] \\
&= \operatorname{sgn}(\alpha', \alpha_i) \left([H_p, X_{\alpha_i}], X_{\alpha'}] + [X_{\alpha_i}, [H_p, X_{\alpha'}]]\right) \\
&= \operatorname{sgn}(\alpha', \alpha_i)(\langle \alpha_i, \alpha_p^\vee \rangle + \langle \alpha', \alpha_p^\vee \rangle)[X_{\alpha_i}, X_{\alpha'}] \\
&= \langle \alpha, \alpha_p^\vee \rangle \operatorname{sgn}(\alpha', \alpha_i)[X_{\alpha_i}, X_{\alpha'}] \\
&= \langle \alpha, \alpha_p^\vee \rangle X_\alpha,
\end{aligned}$$

which completes the proof of (iii).

To prove (vi), we write $\alpha = \alpha' + \alpha_i$ as before, so that $\langle \alpha, \alpha_i^\vee \rangle = 1$ by Lemma 3.6. Let $I, L, L' \in \mathcal{L}_\alpha$ be as in the statement, contrary to hypothesis. By the inductive hypothesis applied to (iii), we have $[H_i, X_{\alpha'}] = X_{\alpha'}$. This gives two possibilities for L (and similar possibilities for L'): the first is that $i \to L$ and but we do not have either $L \leftarrow i$, $L \to i$ or $i \leftarrow L$, and the second is that $L \leftarrow i$ but we do not have either $i \to L$, $i \leftarrow L$ or $L \to i$. Let us consider the first possibility. Because $\chi(L) = \chi(L')$, L' must contain an element labelled i, and since $L \cup L'$ is convex (as $I \cup L$ and $I \backslash L'$ are ideals), we must have $L' \to i$. This is not a permissible configuration for L', so we have a contradiction and we conclude that in fact $L \leftarrow i$. A dual analysis shows that $i \to L'$. If we now delete the maximal element in L with label i to form a heap L_0, and we delete the minimal element in L' with label i to form a heap L'_0, then the inductive hypothesis applied to the ideal I and the heaps $L_0, L'_0 \in \mathcal{L}_{\alpha'}$ shows that this situation is impossible, proving (vi).

It follows from (vi) that $[X_\alpha, Y_\alpha] = H_\alpha$, and we know from [**12**, §7.8] that $[E_\alpha, E_{-\alpha}] = -\alpha^\vee$, so we have $H_\alpha = \alpha^\vee$ by (ii) and (iv). The other assertions follow from [**12**, (5.1.1), §7.8], using the fact that all roots have the same length. This establishes (vii). $\square$

Corollary 5.5. *Let E be a full heap over the Dynkin diagram Γ of A, and let A_0 be the corresponding finite type generalized Cartan matrix. If α, β, γ are positive roots associated to A_0 such that $\alpha = \beta + \gamma$, then any root heap $L \in \mathcal{L}_\alpha$ decomposes uniquely as a disjoint union $L = L_1 \cup L_2$ of (convex) subheaps $L_1 \in \mathcal{L}_\beta$ and $L_2 \in \mathcal{L}_\gamma$ such that one of L_1 and L_2 is an ideal of L and the other is a filter of L.*

Proof. We can choose a proper ideal I of E such that $L \prec I$ by Lemma 2.1 (vii). We have $H_\alpha(v_I) = v_I$, and by Proposition 5.4 (vii), we have $H_\alpha = H_\beta + H_\gamma$. By Proposition 5.4 (vi), there are two ways this can happen: either $H_\beta(v_I) = v_I$ and $H_\gamma(v_I) = 0$, or vice versa. In the first case, there is a filter L_1 of L and an ideal L_2 of L with $L_1 \in \mathcal{L}_\beta$ and $L_2 \in \mathcal{L}_\gamma$, and in the second case, there is an ideal L_1 of L and a filter L_2 of L with $L_1 \in \mathcal{L}_\beta$ and $L_2 \in \mathcal{L}_\gamma$. $\square$

§6. The non simply laced case

The methods presented for simply laced Lie algebras can be generalized to the non simply laced case. The right way to consider this seems to be to regard the non simply laced objects as folded versions of their simply laced untwisted affine counterparts. For roots, this is the procedure described in [**12**, §7.9]. For our purposes, we also need a categorified version of this phenomenon suitable for full heaps.

Let A be a simply laced generalized Cartan matrix of untwisted affine type and let Γ be the corresponding Dynkin diagram, and suppose that μ is a nonidentity graph automorphism of Γ. Although in general μ can be of order 2 or 3, we will make two additional assumptions about μ: (a) μ has order precisely 2 and (b) for any vertex p, $\mu(p)$ and p are not distinct adjacent vertices.

The group $\{1, \mu\}$ acts on the Dynkin diagram Γ, and we denote the orbit containing the vertex p by $f(p) = \bar{p}$. This induces an action on the simple roots α_i, and we extend this to a linear action on $k \otimes_{\mathbb{Z}} R$ by $\mu(a_i\alpha_i) = a_i\mu(\alpha_i)$. Let us also define $f(\alpha_i) = (\alpha_i + \mu(\alpha_i))/2$ and extend linearly to $k \otimes_{\mathbb{Z}} R$.

Let A and Γ be as above and Δ be the set of roots for A. It is known [**12**, Proposition 7.9] that the set $\{f(\alpha) : \alpha \in \Delta\}$ is a root system for a Kac–Moody algebra $\bar{\mathfrak{g}}$ with simple roots $\{f(\alpha_i)\}$. A root $f(\alpha)$ is called *long* if $\alpha = \mu(\alpha)$, and *short* otherwise. The Dynkin diagram $\overline{\Gamma}$ for $\bar{\mathfrak{g}}$ has vertices labelled by the orbits $\bar{p}$, and is such that if p and q are distinct vertices of Γ, then p and q are adjacent in Γ if and only if the (distinct) vertices $\bar{p}$ and $\bar{q}$ are adjacent in $\overline{\Gamma}$. If Γ contains three vertices p, $\mu(p)$ and q such that q is adjacent to both p and $\mu(p)$, then we join $\bar{p}$ and $\bar{q}$ in $\overline{\Gamma}$ by a double edge with an arrow pointing towards $\bar{p}$. (It is possible for this procedure to result in a double edge with two arrows in opposite directions.)

We will say that A (respectively, Γ) *folds* to $\overline{A}$ (respectively, $\overline{\Gamma}$) via μ.

Proposition 6.1. *Let $E = (E, \leq, \varepsilon)$ be a full (labelled) heap over the Dynkin diagram Γ of A, where A is a simply laced generalized Cartan matrix of untwisted affine type. Suppose that μ folds A and Γ to $\overline{A}$ and $\overline{\Gamma}$, and also that whenever we have vertices p, q of Γ satisfying (a) $\mu(p)\ \mathcal{C}\ q$, (b) $\alpha \in \varepsilon^{-1}(p)$ and (c) $\beta \in \varepsilon^{-1}(q)$, then α and β are comparable in E. Then $\overline{E} = (E, \leq, f \circ \varepsilon)$ is a full (labelled) heap over $\overline{\Gamma}$.*

Proof. We first show that $\overline{E}$ is a heap, i.e., that Definition 1.1 holds.

For part 1 of the definition, it is enough to show that if $\alpha, \beta \in E$, then α, β are comparable if $f \circ \varepsilon(\alpha) = f \circ \varepsilon(\beta)$. There are four cases to consider: either $\varepsilon(\alpha) = \varepsilon(\beta)$, or $\varepsilon(\alpha) = \mu(\varepsilon(\beta))$, or $\varepsilon(\alpha)$ is adjacent to $\varepsilon(\beta)$, or $\varepsilon(\alpha)$ is adjacent to $\mu(\varepsilon(\beta))$. In each case, α and β are guaranteed to be comparable either by the definition of a heap, or by the hypotheses on μ in the statement.

For part 2, suppose that $\alpha, \beta \in E$, $\alpha \leq \beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$. It is immediate that $f(\varepsilon(\alpha))\ \mathcal{C}\ f(\varepsilon(\beta))$, from which the assertion follows.

We next show that $\overline{E}$ is fibred. Part (a) of Definition 1.6 comes from the fact that $(f \circ \varepsilon)^{-1}(\bar{p}) \subseteq \varepsilon^{-1}(p)$. For (b), assume that p' and q' are adjacent vertices of

Γ and that $\alpha \in E$ satisfies $f(\varepsilon(\alpha)) = p'$. The properties of graph automorphisms guarantee the existence of a vertex q of Γ adjacent to $p = \varepsilon(\alpha)$ with $f(q) = q'$. Since E is fibred, there exists $\beta \in E$ such that α covers β or β covers α, with $\varepsilon(\beta) = q$. This vertex satisfies $f(\varepsilon(\beta)) = q'$, as required.

Finally, we show that $\overline{E}$ is full. Let $\alpha, \beta \in E$ (where $\alpha < \beta$) be such that $f \circ \varepsilon(\alpha) = f \circ \varepsilon(\beta) = \bar{p}$ and $(\alpha, \beta) \cap (f \circ \varepsilon)^{-1}(\bar{p}) = \emptyset$.

Suppose first that $\varepsilon(\alpha) = \varepsilon(\beta) = p$, say. In this case, the interval (α, β) in E contains precisely two elements, γ, γ', with labels adjacent to p. This shows that the elements γ and γ', considered as elements of $\overline{E}$, are the only two elements of (α, β) with labels adjacent to $\bar{p}$. It remains to show that the Dynkin diagram $\overline{\Gamma}$ does not contain an arrow from $f \circ \varepsilon(\gamma)$ or $f \circ \varepsilon(\gamma')$ to $\bar{p}$; we deal with the former case, the other being similar. Now either β covers γ, or γ covers α (or possibly both); we prove the former, and the latter follows by a dual argument. If such an arrow exists, there must exist β' with $\varepsilon(\beta') = \mu(\varepsilon(\beta))$ and either β' covers γ or γ covers β. If β' covers γ, this implies that β and β' are comparable, contrary to the hypothesis on μ. On the other hand, if γ covers β' then by hypothesis, β' and α are comparable, which forces $\alpha < \beta' < \beta$. This in turn implies that

$$\beta' \in (\alpha, \beta) \cap (f \circ \varepsilon^{-1})(\bar{p}),$$

a contradiction.

The other case to consider is that $\varepsilon(\alpha) \neq \varepsilon(\beta)$, which implies that $\mu(\varepsilon(\alpha)) = \varepsilon(\beta)$. If $\alpha = E(p, t)$ in the numbering of Remark 1.7, define $\alpha' = E(p, t + 1) > \alpha$. By the hypothesis on μ, α' and β are comparable, which (by the hypotheses on α and β) means that $\alpha < \beta < \alpha'$ and that (β, α') is nonempty. Since the interval (α, α') in E contains precisely two elements, γ and γ', with labels adjacent to $\varepsilon(\alpha)$, and neither label is $\varepsilon(\beta)$, we may assume without loss of generality that $\gamma \in (\alpha, \beta)$ and $\gamma' \in (\beta, \alpha')$. This means that the interval (α, β) in $\overline{E}$ contains precisely one vertex, γ, with label adjacent to $\bar{p}$. It remains to check that there is an arrow in the Dynkin diagram from $f \circ \varepsilon(\gamma)$ to $\bar{p}$, and this follows from that fact that β covers γ and γ covers α. $\square$

Remark 6.2. The words "labelled" may be dropped from the statement of Proposition 6.1 using a familiar argument. In the situation of the proposition, we will say that E *folds to* $\overline{E}$.

All the examples we know of full heaps over non simply laced Dynkin diagrams for affine Kac–Moody algebras are obtained from the simply laced Dynkin diagrams by the folding procedure just described. (See the Appendix for details.)

Definition 6.3. Suppose that $E = (E, \leq, \varepsilon)$ is a full heap over the Dynkin diagram Γ of A, where A is a simply laced generalized Cartan matrix of untwisted affine type, and that μ is a diagram automorphism of Γ that folds the triple (A, Γ, E) to $(\overline{A}, \overline{\Gamma}, \overline{E})$. An orientation of Γ is said to be *compatible* with μ if $\mathrm{sgn}(p, p') = \mathrm{sgn}(\mu(p), \mu(p'))$. If L is a finite subheap of $\overline{E}$, and Γ has an orientation compatible with μ, then we define $\overline{L}$ to be the subheap of $\overline{E}$ corresponding to L, and we define $\mathrm{sgn}(\overline{L}) = \mathrm{sgn}(L)$, where parity is taken with respect to this compatible orientation. (Every finite subheap of $\overline{E}$ arises in this way.)

The operators X_α and Y_α in the non simply laced case are now defined in the same way as in Definition 4.5. (The arrows induced by the orientation have nothing to do with the arrows used in the definition of the Dynkin diagram.)

Example 6.4. Let Γ be the Dynkin diagram of type $A_5^{(1)}$, and let E be the full heap over Γ shown in Figure 4 of the Appendix. Let $\overline{E}$ be the corresponding heap over the Dynkin diagram $\overline{\Gamma}$ of type $C_3^{(1)}$ shown in Figure 6. Suppose the Dynkin diagrams are oriented as in Figure 1, and let F be a convex subheap of E with character $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ (all such subheaps are isomorphic). Then the subheap $\overline{F}$ of $\overline{E}$ corresponding to F has character $\alpha_1 + 2\alpha_2 + \alpha_3$. (See Figure 2.) There are two pairs (α, β) of elements in F such that $\alpha > \beta$ and either $\mathrm{sgn}(\varepsilon(\alpha), \varepsilon(\beta)) = -1$ or $\varepsilon(\alpha) = \varepsilon(\beta)$. These are the pairs (α, β) and (γ, β), where $\varepsilon(\alpha) = 5$, $\varepsilon(\beta) = 4$ and

$\varepsilon(\gamma) = 3$. We thus have $\operatorname{sgn}(F) = (-1)^2 = 1$, and we have $\operatorname{sgn}(\overline{F}) = \operatorname{sgn}(F)$ by definition.

FIGURE 1. Compatible orientations for the Dynkin diagrams of types $A_5^{(1)}$ and $C_3^{(1)}$ in Example 6.4

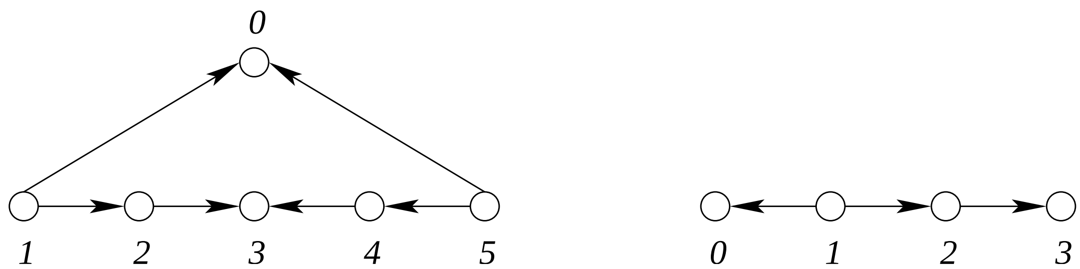
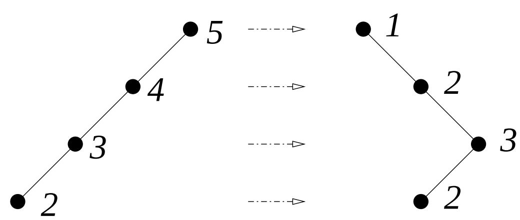


FIGURE 2. The heaps F and $\overline{F}$ of Example 6.4



It is immediate from the definitions that if E folds to $\overline{E}$ via μ and $L \in \mathcal{L}_\alpha$ is a root heap in E, then we have $\overline{L} \in \mathcal{L}_{f(\alpha)}$. The following result shows that the converse is also true, so that we may pass easily between the root heaps of E and those of $\overline{E}$.

Proposition 6.5. *Let E be a full heap over a simply laced Dynkin diagram Γ of untwisted affine type, A, let A_0 be the corresponding finite type generalized Cartan matrix with Kac–Moody algebra $\mathfrak{g}_0$, and let $\alpha = \sum \lambda_i \alpha_i$ be a positive real root associated to $\mathfrak{g}_0$. Suppose that the map f sends the roots of $\mathfrak{g}_0$ to the roots of another Kac–Moody algebra $\overline{\mathfrak{g}_0}$ of finite type, identifying simple roots with simple roots, and that E folds to $\overline{E}$ via μ. Let $f(\alpha)$ be the root of the simple Lie algebra $\overline{\mathfrak{g}_0}$ corresponding to α, and assume that the field k does not have characteristic* 2*.*

(i) *The root $f(\alpha)$ is representable in $\overline{E}$, and for any root heap $\overline{L} \in \mathcal{L}_{f(\alpha)}$, we have $L \in \mathcal{L}_\alpha \cup \mathcal{L}_{\mu(\alpha)}$.*

(ii) *The operator* X_α *is nonzero, lies in the Lie algebra generated by the operators* X_p *on* $V_{\overline{E}}$*, and (in the case* $k = \mathbb{C}$*) is equal to the element* E_α *in the notation of* [**12**, (7.9.3)].

(iii) *The operator* Y_α *is nonzero, lies in the Lie algebra generated by the operators* Y_p *on* $V_{\overline{E}}$*, and (in the case* $k = \mathbb{C}$*) is equal to the element* $-E_{-\alpha}$ *in the notation of* [**12**, (7.9.3)].

Proof. As in Proposition 5.4, we prove the statements by simultaneous induction on $\mathrm{ht}(f(\alpha))$, calculated with respect to the basis of simple roots $f(\alpha_i)$. The case where α is simple follows from the definitions.

Suppose now that $f(\alpha) \neq f(\alpha_i)$ for any simple root α_i. It follows from the definitions that the set $\mathcal{L}_{f(\alpha)}$ is nonempty: we may choose $L_1 \in \mathcal{L}_\alpha$ by Proposition 5.4 (i), and then $\overline{L_1} \in \mathcal{L}_{f(\alpha)}$.

Now let $\overline{L} \in \mathcal{L}_{f(\alpha)}$ be arbitrary. Since $\overline{E}$ is full by Proposition 6.1, we may apply Lemma 5.2 to find a simple root $f(\alpha_i)$ such that $\langle f(\alpha), f(\alpha_i)\rangle > 0$; we may then apply Lemma 5.3 to $\overline{E}$. Let I be a proper ideal of E with $L \succ I$; this exists by Lemma 2.1 (vii).

Suppose that we are in case (i) of Lemma 5.3, so that $\overline{L}$ has a maximal vertex β and a minimal vertex β' with $f \circ \varepsilon(\beta) = f \circ \varepsilon(\beta') = \overline{i}$. Since we are in case (i), we have

$$s_{\overline{i}}(f(\alpha)) = f(\alpha) - 2f(\alpha_i) = f(\alpha) - \alpha_i - \mu(\alpha_i) = f(\alpha'),$$

for some root α'. This means that $\overline{L}\backslash\{\beta, \beta'\} \in \mathcal{L}_{f(\alpha')}$, which by the inductive hypothesis shows that $L\backslash\{\beta, \beta'\} \in \mathcal{L}_{\alpha'}$. We cannot have $\varepsilon(\beta) = \varepsilon(\beta')$, as this would contradict Lemma 3.3, Lemma 3.6 and Proposition 5.4 (iii) applied to $L\backslash\{\beta, \beta'\}$. It must therefore be the case that $p = \varepsilon(\beta) = \mu(\varepsilon(\beta')) = q$, where $\mu(\varepsilon(\beta))$ and $\varepsilon(\beta)$ are distinct. Now consider the element of V_E given by

$$[X_p, [X_q, X_{\alpha'}]].v_I.$$

The bracketed expression expands to

$$X_p \circ X_q \circ X_{\alpha'} - X_p \circ X_{\alpha'} \circ X_q - X_q \circ X_{\alpha'} \circ X_p + X_{\alpha'} \circ X_q \circ X_p.$$

Since μ is compatible with E, any elements of E with labels p and q are comparable, but since p and q are not adjacent, we have $X_p \circ X_q = X_q \circ X_p = 0$. Since L has no minimal element labelled p, we must have $X_p.v_I = 0$. It follows that

$$[X_p, [X_q, X_{\alpha'}]].v_I = -X_p \circ X_{\alpha'} \circ X_q.v_I,$$

which is a nonzero multiple of $v_{I \cup L}$. By Proposition 5.4 (ii) and the properties of the Chevalley bases given in [**12**, (7.8.5)], we see that $[E_p, [E_q, E_{\alpha'}]]$ must be a nonzero multiple of $E_{\alpha' + \alpha_i + \mu(\alpha_i)}$, so that in particular $\alpha' + \alpha_i + \mu(\alpha_i)$ is a root and $L \in \mathcal{L}_\alpha$, proving (i). (A similar argument shows that $\alpha' + \alpha_i$ and $\alpha' + \mu(\alpha_i)$ are both roots.) To prove (ii), we note that $f(\alpha_i)$ is a short root; furthermore, because $\alpha' + \alpha_i$ and $\alpha' + \mu(\alpha_i)$ are roots, $\mu(\alpha') + \alpha_i$ must also be a root. By [**12**, (7.9.6)], if γ and γ' are short roots whose sum is a root, then $\gamma + \mu(\gamma')$ is not a root. It follows that α' must be a long root, which by the choice of orientation on Γ means that

$$\mathrm{sgn}(\alpha_i, \alpha' + \mu(\alpha_i)) = \mathrm{sgn}(\mu(\alpha_i), \alpha' + \alpha_i).$$

Applying Lemma 4.6 now shows that $\mathrm{sgn}(L) = -\mathrm{sgn}(L \backslash \{\beta, \beta'\})$. As operators on $V_{\overline{E}}$, we have

$$\begin{aligned} [X_{\bar{p}}, [X_{\bar{p}}, X_{f(\alpha')}]].v_I &= -2X_{\bar{p}} \circ X_{f(\alpha')} \circ X_{\bar{p}}.v_I \\ &= 2X_{f(\alpha)}.v_I, \end{aligned}$$

where we have used the fact that $X_{\bar{p}} \circ X_{\bar{p}}$ is zero. Since every element of $\mathcal{L}_{f(\alpha)}$ has a maximal and a minimal element labelled $\bar{p}$, we see that

$$[X_{\bar{p}}, [X_{\bar{p}}, X_{f(\alpha')}]] = 2X_{f(\alpha)}.$$

A similar calculation using the Chevalley basis [**12**, (7.9.3)] shows that

$$[E_{\bar{p}}, [E_{\bar{p}}, E_{f(\alpha')}]] = 2\,\mathrm{sgn}(\alpha_i, \alpha')\,\mathrm{sgn}(\mu(\alpha_i), \alpha' + \alpha_i) E_{f(\alpha)} = 2E_{f(\alpha)},$$

thus proving (ii).

The other possibility is that we are in case (ii) of Lemma 5.3, so that

$$s_{\bar{i}}(f(\alpha)) = f(\alpha) - f(\alpha_i) = f(\alpha').$$

We will deal with the subcase where $\overline{L}$ has a minimal vertex β labelled $\bar{p}$, but no such maximal vertex. Let us assume that $\varepsilon(\beta) = p$. A similar, but easier, argument establishes that $\overline{L}\backslash\{\beta\} \in \mathcal{L}_{f(\alpha')}$ and $L\backslash\{\beta\} \in \mathcal{L}_{\alpha'}$. If $\overline{L}$ has a maximal vertex labelled $\bar{p}$ but no such minimal vertex, a similar argument holds. Operating on V_E, we then find (by acting both sides on a suitable v_I) that

$$[X_p, X_{\alpha'}] = \operatorname{sgn}(\alpha_p, \alpha')X_\alpha$$

by Lemma 4.6, Proposition 5.4 (ii) and [**12**, (7.8.5)], from which (i) follows. Analogous calculations on $V_{\overline{E}}$ then show that

$$[X_{\bar{p}}, X_{f(\alpha')}] = \operatorname{sgn}(\alpha_p, \alpha')X_{f(\alpha)},$$

proving (ii).

The proof of (iii) follows by symmetric arguments. In the first case above, this results in the identities

$$[Y_{\bar{p}}, [Y_{\bar{p}}, Y_{f(\alpha')}]] = 2Y_{f(\alpha)}$$

and

$$[E_{\bar{p}}, [E_{\bar{p}}, E_{f(\alpha')}]] = 2E_{f(\alpha)}.$$

In the other case, we obtain

$$[Y_{\bar{p}}, Y_{f(\alpha')}] = \operatorname{sgn}(\alpha', \alpha_p)Y_{f(\alpha)} = -\operatorname{sgn}(\alpha_p, \alpha')Y_{f(\alpha)}.$$

□

Definition 6.6. Let A be a generalized Cartan matrix of untwisted affine type with Dynkin diagram Γ. If either

(i) A is simply laced and E is any full heap over Γ, or

(ii) A is not simply laced and occurs as a matrix $\overline{A}$ arising from a folded heap $E = \overline{E'}$ as in Proposition 6.1,

then we call E a *simply folded* full heap over Γ.

When restricted to the simply laced case, the following theorem is similar to the unproven [**22**, Theorem 4.1].

Theorem 6.7. *Let E be a simply folded full heap over the Dynkin diagram Γ of the generalized Cartan matrix A of an untwisted affine Kac–Moody algebra.*

Let A_0 be the corresponding finite type generalized Cartan matrix with Kac–Moody algebra $\mathfrak{g}_0$ and set of positive roots Δ^+. Then the set of operators

$$\{X_\alpha : \alpha \in \Delta^+\} \cup \{Y_\alpha : \alpha \in \Delta^+\} \cup \{H_p : p \text{ is a vertex of } \Gamma\}$$

on V_E over the field $k = \mathbb{C}$ is linearly independent and its span is isomorphic to the (simple) Lie algebra $\mathfrak{g}_0$; in particular, the isomorphism type depends only on $\mathfrak{g}_0$ (rather than E).

Proof. From [**12**, §7.8], we know that, over $k = \mathbb{C}$, the algebra $\mathfrak{g}_0$ has a basis given by

$$\{E_\alpha : \alpha \in \Delta\} \cup \{E_{-\alpha} : \alpha \in \Delta\} \cup \{\alpha_p^\vee : p \text{ is a vertex of } \Gamma\}.$$

The conclusion now follows from Theorem 3.1, Proposition 5.4 (ii) and (iv) (in the simply laced case) and Proposition 6.5 (ii) and (iii) (in the non simply laced case). □

This theorem can be used to construct all finite dimensional simple Lie algebras over $\mathbb{C}$ except those of types E_8, F_4 and G_2. As we explain in §9, for the simple Lie algebras other than these three, it is possible to perform the construction using a finite dimensional subspace of V_E, and this leads to combinatorial constructions of the spin modules in types B and D without using Clifford algebras. (The type D construction has already been described without proof by Wildberger [**22**].)

§7. Loop algebras and periodic heaps

Having concentrated on the case of Kac–Moody algebras of finite type, we now turn our attention to the corresponding affine algebras. For this purpose, it is convenient to introduce the notion of a periodic heap.

Definition 7.1. Let $(E, \leq, \varepsilon)$ be a locally finite labelled heap over a graph Γ. We call the labelled heap $(E, \leq, \varepsilon)$, and the associated heap $[E, \leq, \varepsilon]$ *periodic* if there exists a nonidentity automorphism $\phi : E \longrightarrow E$ of labelled posets such that $\phi(x) \geq x$ for all $x \in E$.

Remark 7.2.

(i) It is immediate that any periodic heap is necessarily infinite.

(ii) The automorphism ϕ above restricts to an automorphism of the chains $\varepsilon^{-1}(p)$ for p a vertex of Γ. By Remark 1.7, this automorphism must be of the form $\phi(E(p, x)) = E(p, x + t_p)$ for some nonnegative integer t_p depending on p but not on the labelling chosen for E, and furthermore, the automorphism ϕ can be reconstructed from the integers t_p. If $\alpha \in R^+$ is such that $\alpha(p) = t_p$, we will say that ϕ is *periodic with period* α. If there is no automorphism ϕ' of E with period α' such that $\alpha = n\alpha'$ with $n > 1$, then we also say that E is periodic with period α.

Example 7.3. In the notation of Example 2.4, the heap E is periodic with period $\chi(F)$.

Lemma 7.4. *Let E be a full heap over the Dynkin diagram Γ of A, where A is a simply laced generalized Cartan matrix of untwisted affine type (see the Appendix for examples, or* [**4**, Appendix]*,* [**12**, §4.8] *for a complete list). Let A_0 be the generalized Cartan matrix of finite type obtained by omitting the row and column* *of A corresponding to the root α_0. Let δ be the smallest positive imaginary root associated to A.*

(i) *The root δ is representable in E.*

(ii) *The heap E is periodic with period δ.*

(iii) *Every positive root is representable in E.*

Proof. From [**12**, Theorem 5.6 (b)], we see that $\delta = \theta + \alpha_0$, where θ is the highest root of α. By [**12**, Theorem 4.8 (c)], $\left\langle \delta, \alpha_p^\vee \right\rangle = 0$ for all vertices p, and since $\langle \alpha_0, \alpha_0^\vee \rangle = 2$, we have $\langle \theta, \alpha_0^\vee \rangle = -2$. Let $L_\theta \in \mathcal{L}_\theta$; this exists by Proposition 5.4 (i). By Proposition 5.4 (iii), we have $[H_0, X_\theta] = -2X_\theta$, which means that there exist vertices β, β' of $E \backslash L_\theta$, both labelled 0, such that $L^+ = L_\theta \cup \{\beta\} \cup \{\beta'\}$ lies in $\mathcal{L}_{\delta+\alpha_0}$ and such that β (respectively, β') is a maximal (respectively, minimal) element of L^+. By removing either β or β' from L, we obtain an element of $\mathcal{L}_\delta$, thus proving (i).

For (ii), let $L_0 \in \mathcal{L}_\delta$, which is a nonempty set by (i). By [**12**, Theorem 4.8 (c)], we see that L_0 contains at least one element with each possible label from Γ. Let α be a maximal element of L_0, and let $p = \varepsilon(\alpha)$. Since $\left\langle \delta, \alpha_p^\vee \right\rangle = 0$, we see from Lemma 5.3 (iii) that we can convert L_0 into an element L' of $\mathcal{L}_\delta$ (with $L' \neq L_0$) by removing α and replacing it by a new minimal vertex with label i. By repeating this procedure once for each element of the original heap L_0, we obtain a heap $L_1 \in \mathcal{L}_\delta$ such that $L_0 \cong L_1$ as heaps, $L_0 \cap L_1 = \emptyset$ and $L_0 \cup L_1 = L_1 \circ L_0$ is convex.

By applying the above construction again to L_1, we obtain a sequence $\{L_i\}_{i \geq 0}$ of disjoint isomorphic heaps $L_i \in \mathcal{L}_\delta$ such that $L_i \cup L_{i+1}$ is convex. Since L contains at least one element with each label, any element α with $\alpha \leq \beta$ for some $\beta \in L$ lies in one of the heaps L_i for $i \geq 0$.

By a dual argument, we can also find a sequence $\{L_i\}_{i \leq 0}$ with analogous properties, such that any element α with $\alpha \geq \beta$ for some $\beta \in L$ lies in some L_i for $i \leq 0$. Since L_0 contains an element with each possible label, the heaps L_k for $k \in \mathbb{Z}$ partition the set E. It follows that there is a nonidentity automorphism $\phi : E \longrightarrow E$

of labelled posets sending L_k to L_{k-1} for any $k \in \mathbb{Z}$. This construction shows that ϕ has period δ. Since $\delta \neq n\delta'$ for any $n > 1$ with $\delta' \in R^+$ (see [**12**, Theorem 4.8 (c)]), we see that E also has period δ, completing the proof of (ii).

We next prove that if γ is a root of A_0, then $\delta - \gamma$ is representable. By Proposition 5.4 (i), there exists $L \in \mathcal{L}_\gamma$. By Lemma 2.1 (vii), there is a proper ideal I with $L \prec I$, and by periodicity, there exists $L' \in \mathcal{L}_\delta$ with $\phi^{-1}(I) = I \backslash L'$. In this case, L is a filter of L' and $L' \backslash L \in \mathcal{L}_{\delta-\gamma}$, as required.

Part (iii) follows from this and the fact (see [**12**, §7.4]) that the roots of A are precisely the elements of R^+ of the form $\gamma + j\delta$, where $j \in \mathbb{Z}$ and γ is a root of A_0. (The root is positive if either $\gamma > 0$ and $j \geq 0$, or $\gamma < 0$ and $j \geq 1$.) □

Lemma 7.5. *Let $E, \Gamma, A, \mu, f, \overline{A}, \overline{\Gamma}$ and $\overline{E}$ be as in Proposition 6.1, and assume in addition that $\overline{A}$ is a generalized Cartan matrix of untwisted affine type. Let δ and $\bar{\delta}$ be the smallest positive imaginary roots associated to A and $\overline{A}$, respectively, and suppose that we have $f(\delta) = \bar{\delta}$.*

(i) *The heap $\overline{E}$ is periodic with period $\bar{\delta}$.*
(ii) *Every positive root is representable in $\overline{E}$.*

Proof. Let A_0 (respectively, $\overline{A}_0$) be the generalized Cartan matrix of finite type obtained by removing the zeroth row and column of A (respectively, $\overline{A}$).

Since $\overline{A}$ is of untwisted affine type, it follows by [**12**, §7.4] that each positive root of $\overline{A}$ is of the form $n\bar{\delta} + \gamma$, where γ is a root for $\overline{A}_0$. By construction of $\overline{A}$, we have $\gamma = f(\gamma')$ for some root γ' of A_0, and since we have $f(\delta) = \bar{\delta}$ by hypothesis, it follows that $f(n\delta + \gamma') = n\bar{\delta} + \gamma$; note that $n\delta + \gamma'$ is positive because f respects positive and negative roots. Since $n\delta + \gamma'$ is representable in E by Lemma 7.4 (iii), part (ii) follows by folding.

Since E is periodic with period δ, it follows that $\overline{E}$ is periodic with period δ', where $f(\delta) = \bar{\delta} = k\delta'$ for some positive integer k. Writing $\bar{\delta} = \sum a_i \bar{\alpha}_i$, we have by [**12**, Theorem 4.8 (c)] that the a_i are relatively prime integers, so we must have $k = 1$ and $\delta' = \bar{\delta}$, as required. □

Definition 7.6. The automorphism ϕ of Lemma 7.4 (ii) and Lemma 7.5 (i) induces a permutation (also denoted ϕ) of the proper ideals of E. We define an invertible linear map $T : V_E \longrightarrow V_E$ by $T = X_\delta$; note that $T^{-1} = Y_\delta$.

If α_0 is the simple root of A such that $\delta = \theta + \alpha_0$, then we define the *height*, $h(I)$ of a proper ideal I of E to be the maximum integer t such that $E(0,t) \in I$.

We define the linear map $D : V_E \longrightarrow V_E$ by $D(v_I) = h(I)v_I$.

Definition 7.7 ([12, §7]). Let $\mathfrak{g}_0$ be a Kac–Moody algebra of finite type over $k = \mathbb{C}$. Then the *loop algebra* $\mathcal{L}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ is defined to be the k-vector space $k[t, t^{-1}] \otimes_k \mathfrak{g}_0$ with Lie bracket defined by $[P \otimes x, Q \otimes y] := PQ \otimes [x, y]$.

As explained in [**12**, §7], a fundamental property of the untwisted affine Kac–Moody algebras over $k = \mathbb{C}$ is that they can be constructed from loop algebras by adding both a one-dimensional centre $\mathbb{C}K$ (using a universal central extension) and an additional derivation d. More precisely, we have

Theorem 7.8 (see [12, §7]). *Let A be a generalized Cartan matrix of untwisted affine type Let A_0 be the corresponding generalized Cartan matrix of finite type, with Lie algebra $\mathfrak{g}_0$ over $k = \mathbb{C}$ and highest root θ. Then the Kac–Moody algebra $\mathfrak{g}$ associated to A is the vector space*

$$\mathcal{L}(\mathfrak{g}_0) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

equipped with the Lie bracket

$$\begin{aligned}[(t^m \otimes x) + \lambda K + \mu d, (t^n \otimes y) + \lambda_1 K + \mu_1 d] =& (t^{m+n} \otimes [x, y]) + \mu n (t^n \otimes y) \\ & - \mu_1 m (t^m \otimes x) + m \delta_{m,-n} (x|y) K,\end{aligned}$$

for $x, y \in \mathfrak{g}_0, \lambda, \mu, \lambda_1, \mu_1 \in \mathbb{C}, m, n \in \mathbb{Z}$. (See [**12**, §2.2] *for the definition of $(x|y) \in \mathbb{C}$.) Given $\epsilon = \pm 1$, the isomorphism may be chosen to identify the subalgebra $\mathfrak{g}_0$ with the subset $1 \otimes \mathfrak{g}_0$ of $\mathcal{L}(\mathfrak{g}_0)$, and to send the Chevalley generators e_0 and f_0 of $\mathfrak{g}$ to $\epsilon t \otimes E_\theta$ and $\epsilon t^{-1} \otimes -E_{-\theta}$ respectively, where the E_α are as given in* [**12**, (7.8.5)].

Proof. The complete argument may be found in [**12**, §7]. The assertion about the $\pm E_{\pm\theta}$ follows from the fact ([**12**, Remark 7.9 (c)]) that ϵE_θ and $\epsilon E_{-\theta}$ are exchanged by the Chevalley involution of $\mathfrak{g}_0$, and that

$$[-\epsilon E_{-\theta}, \epsilon E_\theta] = -\theta^\vee$$

(see [**12**, (7.8.5), (7.9.3)]). □

Lemma 7.9. *Let A be a generalized Cartan matrix of untwisted affine type with Dynkin diagram Γ, and let E be a simply folded full heap over Γ. Let A_0 be the generalized Cartan matrix of finite type obtained by omitting the row and column of A corresponding to the root α_0, and let $\mathfrak{g}_0$ be the Lie algebra of A_0, identified with the Lie algebra of operators $\mathfrak{g}_E$ on V_E by Theorem 6.7.*

(i) *The map $T : V_E \longrightarrow V_E$ commutes with the operators X_L, Y_L and H_L.*
(ii) *The loop algebra $\mathcal{L}(\mathfrak{g}_E)$ over $\mathbb{C}$ acts faithfully on V_E via $(t^j \otimes P)(v_I) = T^j \circ P(v_I)$.*
(iii) *We have $[D, T^j \circ P] = jT^j \circ P$ in the Lie algebra of operators $\mathcal{L}(\mathfrak{g}_E) \oplus kD$.*

Proof. For part (i), we see that T and T^{-1} commute with H_L by the fact that E is periodic of period δ (Lemma 7.4 (ii) and Lemma 7.5 (i)). Again, by periodicity, we have $X_L.v_I \neq 0$ if and only if $T \circ X_L.v_I \neq 0$ if and only if $X_L \circ T.v_I \neq 0$. By Lemma 4.6, we see that

$$T \circ X_L = \operatorname{sgn}(\delta, \theta) X_{\delta+\alpha}$$

and

$$X_L \circ T = \operatorname{sgn}(\theta, \delta) X_{\delta+\alpha}.$$

By [**12**, (6.2.4), (7.8.3)], we have $\operatorname{sgn}(\delta, \theta) = \operatorname{sgn}(\theta, \delta)$, and it follows that T (and therefore T^{-1}) commutes with the X_L. A similar argument shows that

$$T^{-1} \circ Y_L = Y_L \circ T^{-1} = \operatorname{sgn}(\delta, \theta) Y_{\delta+\alpha},$$

establishing the claim for the Y_L.

To prove (ii), we note that $P(v_I)$ is a linear combination of basis elements v_J with $h(J) = h(I)$ (where h is as in Definition 7.6), but that $T^j(v_I) = \pm v_{\phi^j(I)}$, where $h(\phi^j(I)) = h(I) + j$, because α_0 occurs in δ with coefficient 1. (The latter follows from [**4**, Proposition 17.2 (ii)].) Since $\mathfrak{g}_0$ is simple and its action on V_E is nontrivial, $\mathfrak{g}_0$ acts faithfully on V_E. Part (ii) now follows from the fact that T has infinite order.

Part (iii) follows from the observations that $D \circ T^j(v_I) = (h(I) + j)T^j(v_I)$ and $T^j \circ D(v_I) = h(I)T^j(v_I)$. $\square$

We can now state the main result of this section.

Theorem 7.10. *Let A be a generalized Cartan matrix of untwisted affine type with Dynkin diagram Γ, and let $\mathfrak{g}$ be the corresponding Kac–Moody algebra. Let E be a simply folded full heap over Γ. Let $\mathfrak{v}$ be the Lie algebra of linear operators on V_E over $k = \mathbb{C}$. Then there is a homomorphism of Lie algebras $\psi : \mathfrak{g} \longrightarrow \mathfrak{v}$ sending the Chevalley generators e_i, f_i (for $0 \leq i \leq n$) to the generators X_i, Y_i respectively, and sending the derivation d to the operator D. The kernel of ψ is precisely $\mathbb{C}K$.*

Proof. Comparing the explicit formula of Theorem 7.8 (with K acting as zero) with the Lie algebra $\mathcal{L}(\mathfrak{g}_E) \oplus \mathbb{C}D$ of Lemma 7.9, and identifying $\mathfrak{g}_0$ with $\mathfrak{g}_E$ as in Theorem 6.7, we obtain a homomorphism from $\mathfrak{g}$ to $\mathcal{L}(\mathfrak{g}_E) \oplus \mathbb{C}D$ with the required kernel, sending the generators e_i, f_i (for $0 < i \leq n$) to X_i and Y_i respectively.

To complete the proof, it is enough to show that we have

$$X_0 = \psi(\epsilon T^{-1} \circ -E_{-\theta}),$$

$$Y_0 = \psi(\epsilon T \circ E_\theta),$$

for some fixed $\epsilon = \pm 1$. By Theorem 6.7 and Proposition 5.4, we know that $\psi(E_\theta) = X_\theta$ and $\psi(-E_{-\theta}) = Y_\theta$. By lemmas 7.4 and 7.5, E is periodic with period δ, and the fact that $\delta = \theta + \alpha_0$ then implies that we have $Y_\theta.v_I \neq 0$ if and only if $X_0.v_I \neq 0$ if and only if $T \circ Y_\theta.v_I \neq 0$. Now suppose that $Y_\theta.v_I \neq 0$, define $L \in \mathcal{L}_\delta$ to be such that $\phi^{-1}(I) \cup L = I$, and define $L_\theta \in \mathcal{L}_\theta$ to be such that $I = I' \cup L_\theta$ with

$Y_\theta.v_I = v_{I'}$. This means that $L = L_\theta \cup \{\alpha\}$, where α is a minimal element of L with label 0. By Lemma 4.4, we find that

$$\operatorname{sgn}(L) = \operatorname{sgn}(\{\alpha\}) \operatorname{sgn}(L_\theta) \operatorname{sgn}(\theta, \alpha) = \operatorname{sgn}(L_\theta) \operatorname{sgn}(\theta, \alpha).$$

The assertion for X_0 then follows by defining $\epsilon = \operatorname{sgn}(\theta, \alpha)$. The assertion for Y_0 is proved similarly, using $\epsilon' = \operatorname{sgn}(\alpha, \theta)$. By [**12**, (6.2.4), (7.8.1), (7.8.3)], it follows that $\epsilon = \epsilon'$, as required. $\square$

§8. Quantum affine algebras, crystals, and the Weyl group action

Let A be an $(l+1) \times (l+1)$ generalized Cartan matrix of affine type corresponding to an untwisted Kac–Moody algebra $\mathfrak{g}$ over $k = \mathbb{C}$. We assume that A is indexed in such a way that the removal of the zeroth row and column of A results in the generalized Cartan matrix for the corresponding Kac–Moody algebra of finite type. The sets Π and $\Pi^\vee$ were defined in §3 for the set $I = \{0, 1, \dots, l\}$. We now extend I by redefining it as $\{0, 1, \dots, l+1\}$, and we extend the sets Π and $\Pi^\vee$ accordingly. The $\mathbb{Z}$-bilinear form is then extended by setting

$$\begin{aligned}
\left\langle \alpha_0, \alpha_{l+1}^\vee \right\rangle &= 1,\\
\left\langle \alpha_i, \alpha_{l+1}^\vee \right\rangle &= 0 \text{ if } 1 \leq i \leq l,\\
\langle \alpha_{l+1}, \alpha_0^\vee \rangle &= 1,\\
\langle \alpha_{l+1}, \alpha_i^\vee \rangle &= 0 \text{ if } 1 \leq i \leq l,\\
\left\langle \alpha_{l+1}, \alpha_{l+1}^\vee \right\rangle &= 0;
\end{aligned}$$

this corresponds to the action of H^* on H described in [**4**, §17.1].

There are several slight variants in the literature of the definition of a quantized affine algebra; ours is based on the one in [**2**]. Let q be an indeterminate. For nonnegative integers $n \geq r$, define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]! = \prod_{i=1}^{n} [i],$$
$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}.$$

Since we are in the untwisted case, [**4**, Proposition 17.2 (ii)] shows that, in the notation of [**2**, §2.2], we have $d = 1$. By [**4**, Proposition 17.9 (a, b)], the symbol q_i in [**2**, §2.2] is equal to q if α_i is a short root, and to q^2 if α_i is a long root.

Definition 8.1. Define the quantum affine algebra U to be the associative algebra with 1 over $\mathbb{Q}(q)$ generated by elements E_i, F_i ($i \in I$), q^h (for $h \in \mathbb{Z}\Pi^\vee$), with defining relations

$$q^0 = 1,$$
$$q^h q^{h'} = q^{h+h'},$$
$$q^h E_i q^{-h} = q^{\langle \alpha_i, h \rangle} E_i,$$
$$q^h F_i q^{-h} = q^{-\langle \alpha_i, h \rangle} F_i,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{p=0}^{b} (-1)^p E_i^{(p)} E_j E_i^{(b-p)} = \sum_{p=0}^{b} (-1)^p F_i^{(p)} F_j F_i^{(b-p)} = 0 \quad \text{for } i \neq j,$$

where q_i is as above, $t_i = q_i^h$, $b = 1 - \langle \alpha_j, \alpha_i^\vee \rangle$, $E_i^{(p)} = E_i^p/[p]!$, and $F_i^{(p)} = F_i^p/[p]!$.

The *weight lattice* P of U is the $\mathbb{Z}$-module $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}\Pi^\vee, \mathbb{Z})$. Since P and $\mathbb{Z}\Pi^\vee$ are free $\mathbb{Z}$-modules of the same finite rank, they are in natural duality.

A U-module M is called *integrable* if

(a) all E_i, F_i ($i \in I$) act locally nilpotently; that is, for each $v \in M$ we have $E_i^N.v = F_i^N.v = 0$ for sufficiently large N, and

(b) M admits a *weight space decomposition*:

$$M = \bigoplus_{\lambda \in P} M_\lambda, \text{ where } M_\lambda = \{u \in M : q^h u = q^{\langle \lambda, h \rangle} u \text{ for all } h \in \mathbb{Z}\Pi^\vee\}.$$

Let A_0 be the subring of $\mathbb{Q}(q)$ consisting of the rational functions of q that are regular at $q = 0$.

Let M be an integrable U module. Kashiwara [**13**] showed that we have

$$M = \bigoplus_{\lambda} F_i^{(n)}(\ker E_i \cap M_\lambda),$$

and defined operators $\tilde{e}_i, \tilde{f}_i : M \longrightarrow M$ for each $0 \leq i \leq l$ (often called *Kashiwara operators*) by

$$\tilde{f}_i(F_i^{(n)} u) = F_i^{(n+1)} u \text{ and } \tilde{e}_i(F_i^{(n)} u) = F_i^{(n-1)} u,$$

where $u \in \ker E_i \cap M_\lambda$. (We interpret $F_i^{(-1)} u = 0$ above.) Any element of M is uniquely expressible as a sum of such elements $F_i^{(n)} u$.

Definition 8.2. Let M be an integrable U-module. A pair $(\mathcal{L}, \mathcal{B})$ is called a *crystal basis* of M if it satisfies:

(i) $\mathcal{L}$ is a free A_0-submodule of M such that $M \cong \mathbb{Q}(q) \otimes_{A_0} \mathcal{L}$,
(ii) $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ for $\lambda \in P$,
(iii) $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{Q} \otimes_{A_0} \mathcal{L}$,
(iv) $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$, $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$,
(v) if we denote operators on $\mathcal{L}/q\mathcal{L}$ induced by $\tilde{e}_i$, $\tilde{f}_i$ by the same symbols, we have $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \sqcup \{0\}$, $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \sqcup \{0\}$,
(vi) for any $b, b' \in \mathcal{B}$ and $i \in I$, we have $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

The following result is a q-analogue of Theorem 7.10 (but without the assertion about the kernel).

Theorem 8.3. *Let A be generalized $(l+1) \times (l+1)$ Cartan matrix of untwisted affine type and let $\mathfrak{g}$ be the corresponding Kac–Moody algebra. Let E be a simply folded full heap over Γ.*

(i) *Over the field $k = \mathbb{Q}(q)$, V_E has the structure of a (left) U-module such that E_i and F_i act as X_i and Y_i, and such that for all $h \in \mathbb{Z}\Pi^\vee$ we have $q^h.v_I = q^\lambda v_I$, where $\lambda \in \mathbb{Z}$ is such that $h.v_I = \lambda v_I$. Here, we identify $\alpha_i^\vee$ with H_i for $0 \leq i \leq l$, and with the operator D for $i = l+1$.*

(ii) *The U-module V_E is integrable and has a crystal basis $(\mathcal{L}, \mathcal{B})$, where*

$$\mathcal{B} = \{v_I : I \text{ is a proper ideal of } E\}$$

and $\mathcal{L}$ is the free A_0-module with $\mathcal{B}$ as a basis.

Proof. Apart from the relations involving $q^{\alpha_{l+1}^\vee}$ and D, part (i) follows by imitating the proof of Theorem 3.1, substituting exponentials where necessary. It follows from the definitions that $[D, X_i] = \delta_{0i} X_i$, and that $[D, Y_i] = -\delta_{0i} Y_i$. The remaining assertions of (i) now follow from the definition of $\langle \alpha_i, \alpha_{l+1}^\vee \rangle$.

The action of E_i and F_i on V_E is locally nilpotent by Lemma 2.7 (7). Since $H_i(v_I)$ and $D(v_I)$ are integer multiples of v_I, it follows that for $h \in \mathbb{Z}\Pi^\vee$, we have

$$q^h.v_I = q^{\lambda(h)} v_I$$

for some $\lambda \in P$ depending on I but not h. This shows that each v_I is a weight vector. Parts (i), (ii) and (iii) of Definition 8.2 now follow from the definition of $\mathcal{L}$.

Now fix generators E_i and F_i, and a basis element v_I. Since E_i^2 acts as zero, and the action of E_i and F_i takes basis elements either to other basis elements, or to zero, we see that either v_I lies in $\ker E_i$, or that $v_I = F_i.v_{I'}$ for some $v_{I'} \in \ker E_i$. Using the definition of raising and lowering operators, we now find that the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$ are simply given by the actions of F_i and E_i respectively. Parts (iv), (v) and (vi) of Definition 8.2 now follow. $\square$

Remark 8.4. Notice that the distributive lattice structure induced by Corollary 2.2 is compatible with the partial order induced on the basis by the Kashiwara operators. It would be interesting to know whether this phenomenon is typical.

Much of the literature about crystals deals with the case of crystals with extremal weight vectors, but the crystals mentioned in the theorem do not have this property. Another possible approach to these crystals would be to bypass the quantum affine algebra and use Kashiwara's notion of *abstract crystals*, which are crystals equipped with formal weight functions, satisfying certain axioms. We do not pursue this here for reasons of space.

Proposition 8.5. *Maintain the assumptions of Theorem 8.3, and assume that* Γ *has finitely many vertices. Then the* U*-module* V_E *is cyclic and is generated by any one basis element* v_I*.*

Proof. Let v_I and v_J be basis elements. It is enough to exhibit an element $u \in U$ such that $u.v_I = v_J$. Let $K = I \cap J$. Since $K \subseteq I$, $I \backslash K$ is finite by Lemma 2.1 (vi), so it follows that there is a finite sequence $j_1, \dots, j_l$ such that

$$F_{j_1} F_{j_2} \cdots F_{j_l}.v_I = v_K.$$

A similar argument shows that there is a finite sequence $i_1, \dots, i_k$ such that

$$E_{i_1} E_{i_2} \cdots E_{i_k}.v_K = v_J.$$

Concatenating these sequences produces the required element u. ☐

In [**14**], Kashiwara introduces the notion of a "normal crystal" (now often referred to as a "regular crystal"; see [**2**, §2.8]). Such crystals naturally carry an action of the associated Weyl group. In this section, we show that the crystals of Theorem 8.3 also have this property, and that furthermore, the action factors through a Temperley–Lieb type quotient. This brings to light some representation theoretic obstructions to finding full heaps for certain Kac–Moody algebras.

Definition 8.6. Let A be a generalized Cartan matrix with Dynkin diagram Γ, and let E be a simply folded full heap over Γ. For each $i \in I$, we define a linear operator S_i on V_E by requiring that

$$S_i(v_I) = \begin{cases} F_i(v_I) & \text{if } F_i(v_I) \neq 0, \\ E_i(v_I) & \text{if } E_i(v_I) \neq 0, \\ v_I & \text{otherwise.} \end{cases}$$

The definition of full heap guarantees that the cases in the above definition do not overlap.

Proposition 8.7. *Suppose that A, E and Γ satisfy the hypotheses of Definition 8.6, and let $W = W(\Gamma)$ be the Weyl group of Γ.*

(i) *The assignment $s_i \mapsto -S_i$ defines a unique (left) kW-module structure on V_E.*

(ii) *If s_i, s_j are a pair of noncommuting generators of W and the subgroup of W they generate, $W_{ij} = \langle s_i, s_j \rangle$, is finite, then the element*

$$\sum_{w \in W_i} w$$

of kW annihilates V_E.

(iii) *The kW-module V_E is cyclic, and any of the basis elements v_I is a generator.*

Proof. To prove (i), we need to check the defining relations of the Weyl group. The relation $s_i^2 = 1$ holds by Theorem 8.3 (ii) and Definition 8.2 (vi). If $a_{ij} = 0$, the relation $s_i s_j = s_j s_i$ follows from the fact that each element of $\{E_i, F_i\}$ commutes with each element of $\{E_j, F_j\}$.

Now let s_i and s_j be a pair of noncommuting generators, and let v_I be a basis element.

Let us first suppose that either $F_i.v_I \neq 0$ or $F_j.v_I \neq 0$. Since a heap cannot have two maximal vertices with adjacent labels, these possibilities are mutually exclusive, so without loss of generality, we may assume that $F_i.v_I \neq 0$. We may now invoke Lemma 2.5 with $p = i$ and $q = j$.

There are five subcases to consider. In case 1 (respectively, 2), we have $s_i s_j s_i = s_j s_i s_j$ and case (i) (respectively, (ii)) of Lemma 2.5 applies. In case 3, (respectively, 4, 5) we have $s_i s_j s_i s_j = s_j s_i s_j s_i$ and case (i) (respectively, (ii), (iii)) of Lemma 2.5 applies. In each case, the Weyl group relation is respected by the claimed module action, and we have $\sum_{w \in W_i} w.v_I = 0$. For example, in case 4, the identity and $S_i S_j S_i$ both act as the identity; S_i and $S_j S_i$ both act as F_i; S_j, $S_i S_j S_i S_j$ and $S_j S_i S_j S_i$ all act as E_j; and $S_i S_j$ and $S_j S_i S_j$ both act as $E_i E_j$.

Now let us suppose that $F_i.v_I = F_j.v_I = 0$. If we also have $E_i.v_I = E_j.v_I = 0$, both S_i and S_j acts as the identity on v_I, and it is clear that the Weyl group relation holds and that $\sum_{w \in W_i} w.v_I = 0$, so we may assume that this is not the case. As before, the conditions $E_i.v_I \neq 0$ and $E_j.v_I \neq 0$ are mutually exclusive, so we may assume that $E_i.v_I \neq 0$ without loss of generality. There must therefore exist a minimal element $\alpha \in E \backslash I$ with $\varepsilon(\alpha) = i$.

In either case, we apply Lemma 2.5 to the ideal $I \cup \{\alpha\}$, with $p = i$ and $q = j$. There are two subcases to consider, according as $s_i s_j s_i = s_j s_i s_j$ or $s_i s_j s_i s_j = s_j s_i s_j s_i$. In either case, part (ii) of the lemma must apply. For example, in the former case, the identity and S_j both act as the identity; S_i and $S_i S_j$ both act as E_i; and $S_j S_i$, $S_i S_j S_i$ and $S_j S_i S_j$ all act as $E_j E_i$. We conclude that the Weyl group relation holds and that $\sum_{w \in W_i} w.v_I = 0$.

This completes the proofs of parts (i) and (ii). Part (iii) follows by the same argument used to prove Proposition 8.5. $\square$

Corollary 8.8. *There are no full heaps over any Dynkin diagrams of finite type, or over those of types $F_4^{(1)}$, $E_8^{(1)}$, or $E_6^{(2)}$.*

Proof. Let J be the ideal of $\mathbb{Z}W$ generated by the elements $\sum_{w \in W_i} w$ for each subgroup W_i of W generated by a pair of noncommuting generators. The algebra $\mathbb{Z}W/J$ is precisely the generalized Temperley–Lieb algebra $TL(W)$ of [**6**], with the parameter q specialized to 1. (The sign twists in Proposition 8.7 (i) were inserted for compatibility of the ideal J with [**6**].)

If the hypotheses of Proposition 8.7 hold, then we may set $k = \mathbb{Q}$, and the definition of full heap implies that V_E is an infinite dimensional cyclic $\mathbb{Q} \otimes_{\mathbb{Z}} TL(W)$-module. This means that $TL(W)$ and its q-analogue must have infinite rank. This cannot happen if A is of finite type, because in this case W is well known to be a finite group. If A is of type $F_4^{(1)}$ (respectively, $E_8^{(1)}$, $E_6^{(2)}$) then the algebra $TL(W)$ is of type F_5 (respectively, E_9, F_5) in the notation of [**6**], and is of finite rank by [**6**, Theorem 7.1]. $\square$

§9. Applications and questions

We now outline how the results of this paper can be used to simplify those described by Wildberger [Wi] and generalize them to the non simply laced case.

Let E be a simply folded heap over the Dynkin diagram Γ of an untwisted affine Kac–Moody algebra, where Γ is labelled so that vertex p_0 is the additional vertex relative to the corresponding finite type algebra, $\mathfrak{g}_0$, and where E is labelled as in Remark 1.7. Let

$$E' = \{x \in E : x \leq E(p_0, 0)\},$$

and let E_0 be the subheap of E consisting of the vertices

$$E_0 = \{x \in E : x \not\geq E(p_0, 1)\} \backslash E',$$

where we regard E' and E_0 as subheaps of E. It is straightforward to check that the map $J \mapsto J \cup E'$ defines an order-preserving bijection between the ideals of E_0 and the proper ideals of E of height zero, so that the ideals of E_0 are in natural bijection with the orbits of proper ideals of E under the action of the automorphism ϕ. This leads to an irreducible representation of the simple Lie algebra $\mathfrak{g}_0$ over $\mathbb{C}$, where the highest and lowest weight vectors correspond to the ideals E_0 and $\emptyset$ of E_0, respectively.

Wildberger's approach is to work directly with the set of ideals of E_0 (but only in the simply laced case), using raising and lowering operators similar to those in this paper. Our approach is simpler in that (a) we do not need to impose a partial order on the set of convex subheaps and (b) the parity of a root heap, in the simply laced case, can be easily computed intrinsically in terms of its isomorphism type, whereas in [Wi] the parity of a root heap in $\mathcal{L}_\alpha$ is computed by comparing it with a canonical representative of $\mathcal{L}_\alpha$. The approach here gives us enough control over the signs that we can make a precise link with the Chevalley bases in [**12**], for each possible orientation of the diagram.

When we apply this technique to the simple Lie algebras of type B, we obtain a combinatorial construction of the spin representation that does not involve Clifford algebras, analogous to the constructions described by Wildberger [Wi, §5] for type D. (The reader is referred to [**4**, §13.5] for the Clifford algebra construction.) In these cases, the finite heap E_0, whose size grows quadratically with the rank, may be much smaller than the dimension of the spin module, which grows exponentially with the rank. The number of ideals of E_0 is a power of 2 in this case, which may be shown by exhibiting a bijection between ideals of E_0 and certain paths; we hope to give details of this elsewhere. It would be interesting to see if the Clifford algebra itself has an action by raising and lowering operators on the spaces V_{E_0} and V_E.

It may be conjectured that the aforementioned representation of the finite dimensional Lie algebra $\mathfrak{g}_0$ will be a minuscule representation, and that there will be a 1–1 correspondence between simply folded full heaps over untwisted affine Dynkin diagrams and minuscule representations of simple Lie algebras over $\mathbb{C}$. Such a result would require a classification of full heaps over untwisted affine Dynkin diagrams. When the Dynkin diagram contains no circuits, i.e., in types other than A, this is relatively easy because the heaps are ranked as posets; the latter may be proved by [**8**, Theorem 2.1.1 (iii)] and has an analogue for minuscule heaps [**20**, Corollary 3.4]. In type $A_l^{(1)}$, things are more complicated, but based on the results of [**24**], we expect that there will be l isomorphism classes, most of which will not be ranked. A good context to examine these might be the extended slant lattices of Hagiwara

[**10**, §8].

The crystal bases for minuscule modules for simple Lie algebras have been known for some time; for example, they are implicit (in the form of canonical bases) in the work of Lusztig [**18**, Theorem 19.3.5, Proposition 28.1.4]. Another construction of these bases may be given by restricting the crystal basis $\mathcal{B}$ arising from a simply folded full heap E to the finite dimensional module corresponding to the finite subheap E_0. An advantage of our approach is that one can describe the action of a Chevalley basis on the canonical basis.

Full heaps also exist over certain infinite Dynkin diagrams, such as the diagrams A_∞, B_∞ and D_∞ of [**12**, §7.11]. In these cases, the heaps arising are reminiscent of those needed to construct spin representations in the finite case. However, we do not know any examples of full heaps over finite graphs that do not correspond to affine Kac–Moody algebras.

In a future paper, we will show that V_E has an interesting structure as a module for the affine Weyl group. In many cases, this gives new, uniform constructions of representations familiar from other contexts: for example, in types $A_l^{(1)}$ and $E_6^{(1)}$ the module structure appears to agree with certain of the author's cell modules for tabular algebras (see [**7**, §6] and [**9**, §1.2]) after specializing the parameter to 1.

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Appendix: Examples of simply folded full heaps

In the appendix, we give some examples of full heaps over Dynkin diagrams of affine Kac–Moody algebras. All these heaps are periodic, and the dashed boxes in the diagrams indicate the repeating motif. (Note that, for an untwisted affine algebra, the number of elements in the dashed box is the Coxeter number of the associated finite type algebra.)

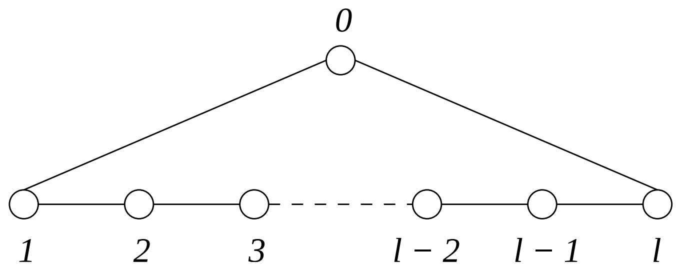
Folding by each of the automorphisms μ shown below either leaves the period of a heap E unchanged, or, in the case of a twisted affine algebra or type $A_1^{(1)}$, the period is halved. In the latter case, the number of orbits of proper ideals of E under the action of ϕ is also halved.

Recall that if $(E, \leq)$ is a partially ordered set, a function $\rho : E \longrightarrow \mathbb{Z}$ is said to be a *rank function* for $(E, \leq)$ if whenever $a, b \in E$ are such that $a < b$ is a covering relation, we have $\rho(b) = \rho(a) + 1$. If a rank function for $(E, \leq)$ exists, we say $(E, \leq)$ is *ranked*. The heaps shown in this section are all ranked.

Type $A_l^{(1)}(l > 1)$, natural representation.

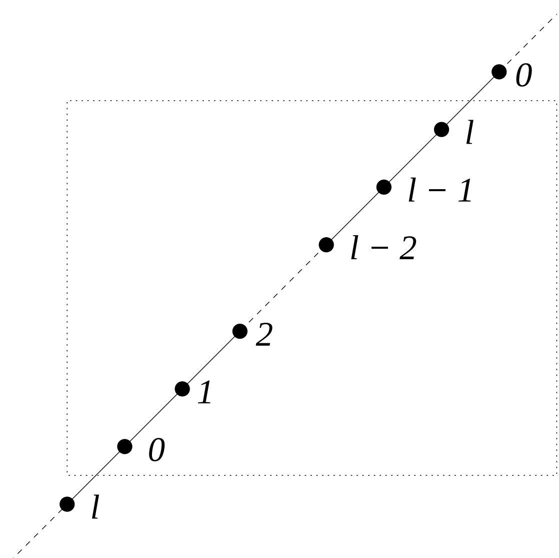
The Dynkin diagram Γ of type $A_l^{(1)}$ (for $l > 1$) is labelled as in Figure 8.

FIGURE 3. The Dynkin diagram of type $A_l^{(1)}(l > 1)$



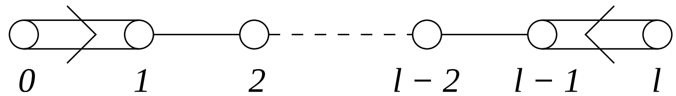
The finite heap E_0 corresponding to the full heap E shown in Figure 4 gives rise to the natural representation of the simple Lie algebra of type A_l (see [**4**, §8.1] for more details). In Wildberger's notation [**22**], we have $E_0 = F(A_l, 1)$. It is clear that the heap E has $l+1$ orbits of proper ideals under ϕ.

FIGURE 4. A full heap, E, over the
Dynkin diagram of type $A_l^{(1)}(l > 1)$



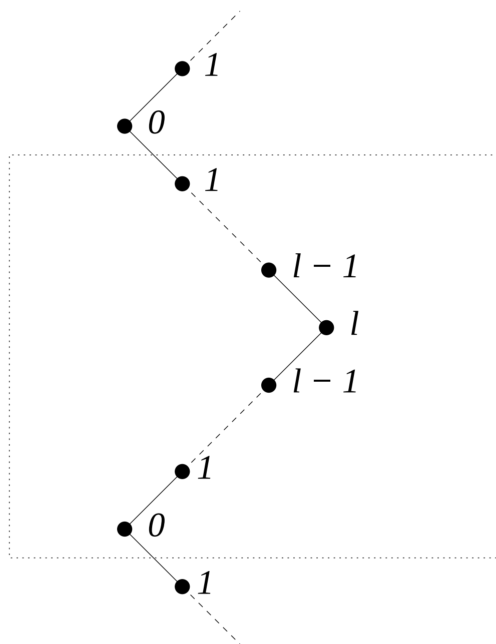
The Dynkin diagram Γ has an automorphism μ given by sending vertex i to vertex $l+1-i$ and fixing vertex 0. If l is odd, then μ also fixes $(l+1)/2$, and we obtain the Dynkin diagram $\overline{\Gamma}$ of type $C_{l'}$, where $l = 2l' - 1$.

FIGURE 5. The Dynkin diagram of type $C_l^{(1)} (l > 1)$



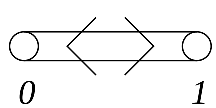
The heap E folds to a heap $\overline{E}$ over $\overline{\Gamma}$, via μ. The heap $\overline{E}$ has $2l$ orbits of proper ideals.

FIGURE 6. A full heap, $\overline{E}$, over the
Dynkin diagram of type $C_l^{(1)} (l > 1)$



The Dynkin diagram of type $A_1^{(1)}$ is as shown in Figure 7.

FIGURE 7. The Dynkin diagram of type $A_1^{(1)}$

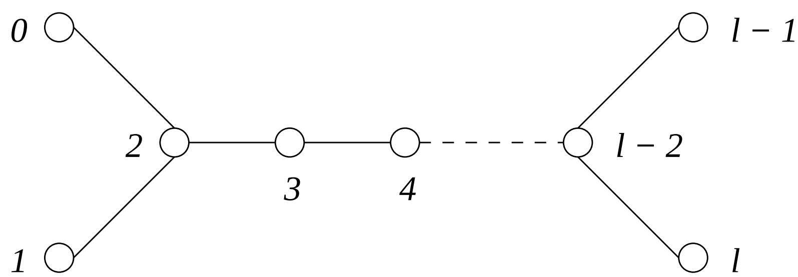


If $l = 3$, so that the Dynkin diagram Γ of type $A_l^{(1)}$ is a square, there is an automorphism μ of Γ given by rotation by a half turn. In this case, the full heap of Figure 4 folds to a full heap over the Dynkin diagram of type $A_1^{(1)}$. (This causes the period to halve, which is behaviour usually characteristic of the twisted affine case.) The results of this paper may be checked by hand for this case.

Type $D_l^{(1)}$, natural representation.

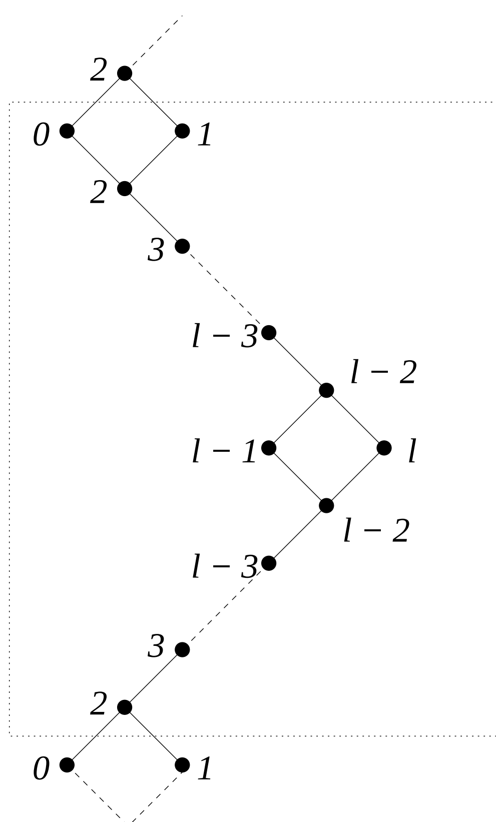
The Dynkin diagram Γ of type $D_l^{(1)}$ is labelled as in Figure 8.

FIGURE 8. The Dynkin diagram of type $D_l^{(1)}$



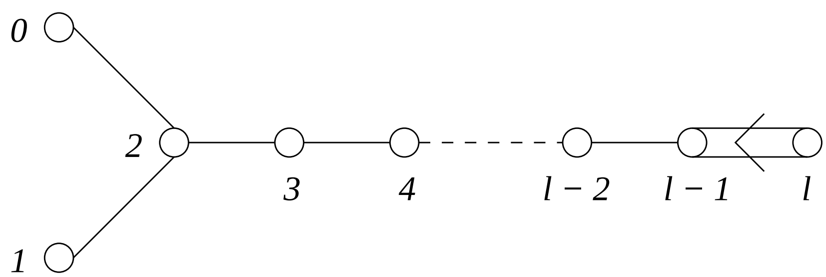
The finite heap E_0 corresponding to the full heap E shown in Figure 9 gives rise to the natural representation of the simple Lie algebra of type D_l; see [**4**, §8.2] for details of another construction of this representation. In Wildberger's notation, we have $E_0 = F(D_l, l-1)$. The heap E has $2l$ orbits of proper ideals under ϕ.

FIGURE 9. A full heap, E, over the Dynkin diagram of type $D_l^{(1)}$



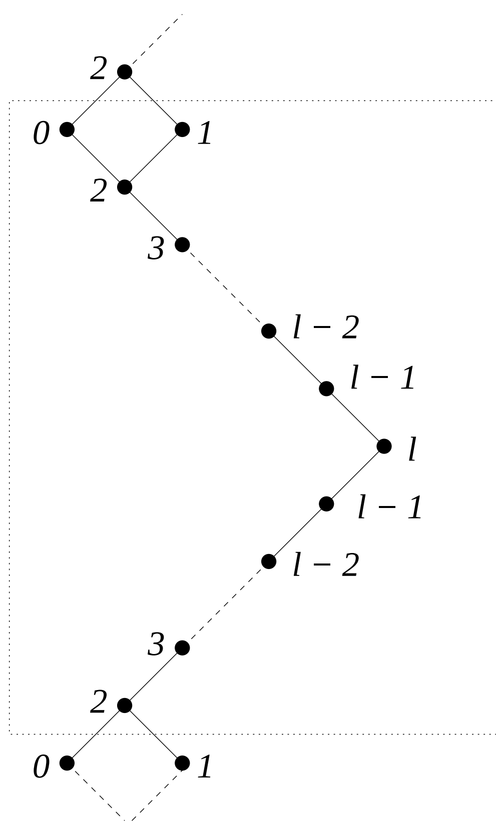
The Dynkin diagram Γ has an automorphism μ given by sending vertex i to vertex $l+1-i$. If l is odd, then μ has a fixed point and we obtain the Dynkin diagram $\overline{\Gamma}$ of type $A^{(2)}_{2l-1}$ from that of type $D^{(1)}_{2l-1}$; the former is shown in Figure 10. The orbit $\{i, 2l-i\}$ in type $D^{(1)}_{2l-1}$ corresponds to the vertex i in type $A^{(2)}_{2l-1}$.

FIGURE 10. The Dynkin diagram of type $A^{(2)}_{2l-1}$



The heap E folds to a heap $\overline{E}$ over $\overline{\Gamma}$, via μ, shown in Figure 11.

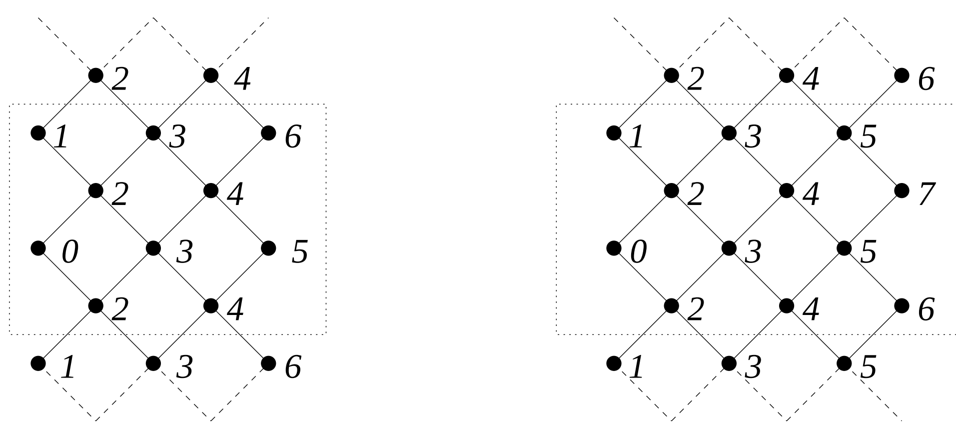
FIGURE 11. A full heap, $\overline{E}$, over the Dynkin diagram of type $A^{(2)}_{2l-1}$



Type $D_l^{(1)}$, spin representations.

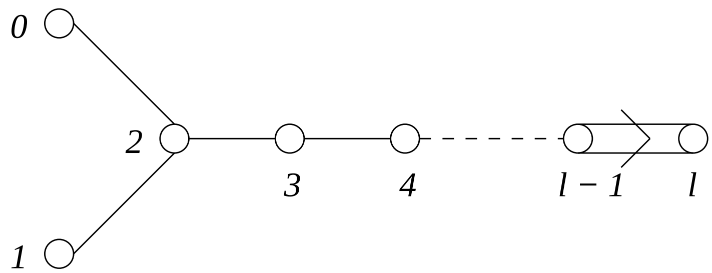
We now consider full heaps E over the Dynkin diagram of type $D_l^{(1)}$ (see Figure 8) corresponding to a spin representation of the simple Lie algebra of type D_l. The heap E is ranked; we will call the subheap of E given by the elements of rank k the "k-th layer" of E. The even numbered vertices in the set $X = \{2, 3, 4, \ldots, l-2\}$ occur in the k-th layer if and only if k is even, and the odd numbered vertices in X occur in the k-th layer if and only if k is odd. If l is odd, then the k-th layer contains precisely one other vertex, labelled $l-1$, 0, l or 1 according as $k = 0$, 1, 2 or 3 mod 4. If l is even, then the $(2k+1)$-st layer contains precisely two other vertices, whose labels are 0 and $l-1$ if k is odd, and 1 and l if k is even. Figure 12 shows examples of such heaps for $l = 6$ and $l = 7$. Another isomorphism class of heaps may be obtained in each case by twisting by the graph automorphism exchanging vertices $l-1$ and l. The corresponding finite heaps E_0 are $F(D_l, 0)$ and $F(D_l, 1)$ in Wildberger's notation. There are 2^{l-1} orbits of proper ideals of E under ϕ.

FIGURE 12. Full heaps over the Dynkin diagram of type $D_l^{(1)}$ for $l = 6$ and $l = 7$



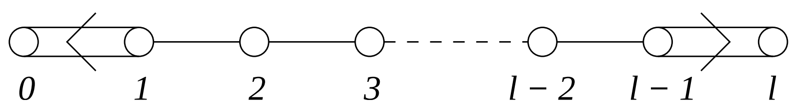
There is an automorphism μ_1 of the Dynkin diagram of $D_l^{(1)}$ obtained by exchanging vertices $l-1$ and l, and fixing each of the other vertices. The full heaps over $D_l^{(1)}$ discussed above fold via μ_1 to a heap over the Dynkin diagram of type $B_{l-1}^{(1)}$ (see Figure 13). The only difference this makes to the full heap is that the vertices formerly numbered l all have their labels changed to $l-1$; this has the effect of merging the two isomorphism classes.

FIGURE 13. The Dynkin diagram of type $B_l^{(1)}$



There is an automorphism μ_2 of the Dynkin diagram of $D_l^{(1)}$ obtained by exchanging 0 with 1, exchanging $l-1$ with l, and fixing each of the other vertices. The full heaps over $D_l^{(1)}$ discussed above fold via μ_2 to a single heap over the Dynkin diagram of type $D_{l-1}^{(2)}$ (see Figure 14). In this case, heap elements formerly labelled 0 or 1 are relabelled by 0, heap elements formerly labelled $l-1$ or l are relabelled by $l-2$, and other heap elements have their labels decreased by 1.

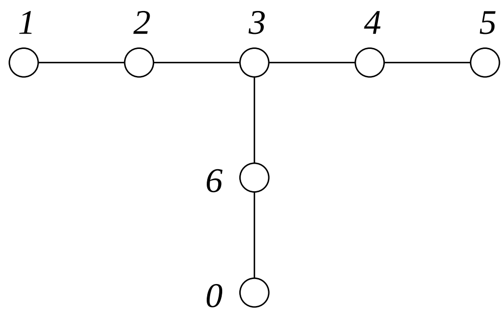
FIGURE 14. The Dynkin diagram of type $D_{l+1}^{(2)}$



Type $E_6^{(1)}$.

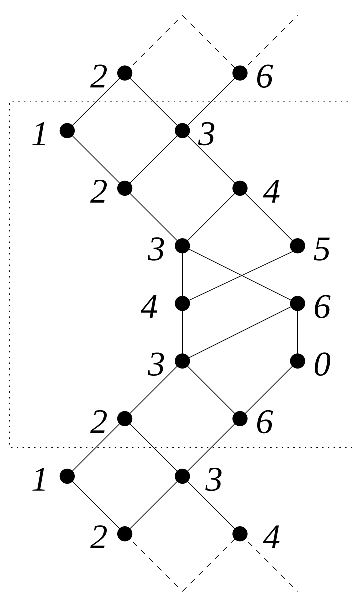
The Dynkin diagram Γ of type $E_6^{(1)}$ is labelled as in Figure 15.

FIGURE 15. The Dynkin diagram of type $E_6^{(1)}$



There are two full heaps over Γ: the one shown in Figure 16, and its dual, which may be constructed by applying by twisting by a diagram automorphism corresponding to an odd permutation to the branches of the Dynkin diagram emerging from vertex 3. The corresponding finite heaps are $F(E_6, 1)$ and $F(E_6, 5)$ in Wildberger's notation. There are 27 orbits of proper ideals of E under ϕ.

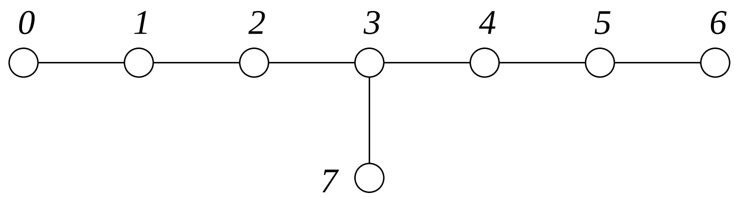
FIGURE 16. A full heap over the Dynkin diagram of type $E_6^{(1)}$



Type $E_7^{(1)}$.

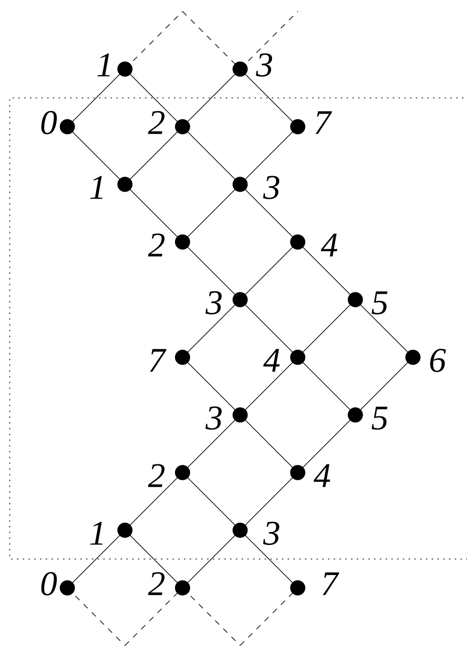
The Dynkin diagram Γ of type $E_7^{(1)}$ is labelled as in Figure 17.

FIGURE 17. The Dynkin diagram of type $E_7^{(1)}$



There is one (self-dual) full heap over Γ, shown in Figure 18. The corresponding finite heap in Wildberger's notation is $F(E_7, 6)$. There are 56 orbits of proper ideals of E under ϕ.

FIGURE 18. A full heap over the Dynkin diagram of type $E_7^{(1)}$



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